String Theory 2007

Tutorial Sheet 6

Superalgebras

The following problems deal with Lie superalgebras, particularly in supergravity and string theories. A great reference for the uses of the superalgebra is [2], which might be useful for some of the problems.

Problem 6.1 Let $\mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{g}_1$ be a Lie superalgebra.

a. Show that $\mathfrak{h}_0 := [\mathfrak{g}_1, \mathfrak{g}_1]$ is an ideal of $\mathfrak{g}_0$, whence $\mathfrak{h} = \mathfrak{h}_0 \oplus \mathfrak{g}_1$ is a superideal of $\mathfrak{g}$.

b. Show that $\dim \mathfrak{h}_0 \le \frac{1}{2} \dim \mathfrak{g}_1 (\dim \mathfrak{g}_1 + 1)$.

c. Let $\dim \mathfrak{h}_0 = \frac{1}{2} \dim \mathfrak{g}_1 (\dim \mathfrak{g}_1 + 1)$, so that the Lie bracket $[,] : S^2 \mathfrak{g}_1 \to \mathfrak{h}_0$ is an isomorphism. Show that relative to a basis Q_a for $\mathfrak{g}_1$, and $Z_{ab} := [Q_a, Q_b]$ for $\mathfrak{h}_0$, the Lie superalgebra $\mathfrak{h}$ has the following structure:

$$[Z_{ab}, Q_c] = \omega_{bc} Q_a + \omega_{ac} Q_b$$
$$[Z_{ab}, Z_{cd}] = \omega_{bc} Z_{ad} + \omega_{ac} Z_{bd} + \omega_{ad} Z_{bc} + \omega_{bd} Z_{ac}$$

for some $\omega_{ab} = -\omega_{ba}$. Show furthermore that the corresponding $\omega \in \Lambda^2 \mathfrak{g}_1^*$ is $\mathfrak{h}_0$-invariant. Is ω $\mathfrak{h}$-invariant?
(*Hint*: If you get stuck, you might want to look at the Appendix of [1].)

Problem 6.2 Consider the eleven-dimensional Poincaré superalgebra

$$[Q_a, Q_b] = \gamma^{\mu}_{ab} P_{\mu} ,$$

where Q_a transforms in the spinor representation Δ of $\mathrm{Spin}(1,10)$.

a. Show that the massless supermultiplet induced from the trivial representation of the little group $\mathrm{Spin}(9)$ corresponds to the massless representation of the Poincaré group induced from the representation $S^2_0 \oplus \Lambda^3 \oplus R$ of $\mathrm{Spin}(9)$, where S^2_0 denotes the symmetric traceless tensors and R is the kernel of the Clifford multiplication $\Lambda^1 \otimes \Delta \to \Delta$, where Δ is the spinor representation of $\mathrm{Spin}(9)$.
(*Hint*: Show that the supermultiplet in question is isomorphic to the irreducible Clifford module $\mathfrak{M}$ of $C\ell(16)$ and then simply decompose $\mathfrak{M}$ under $\mathrm{Spin}(9)$ using the (maximal) embedding $\mathfrak{spin}(9) < \mathfrak{spin}(16)$.)

b. Interpret the resulting representations in terms of eleven-dimensional fields.

c. Show that the massless supermultiplet induced from a finite-dimensional representation V of $\mathrm{Spin}(9)$ corresponds to the massless representation of the Poincaré group induced from the representation $(S^2_0 \oplus \Lambda^3 \oplus R) \otimes V$ of $\mathrm{Spin}(9)$.

Problem 6.3 Consider the **M-superalgebra** introduced in the lecture:

$$[Q_a, Q_b] = \gamma^{\mu}_{ab} P_{\mu} + \tfrac{1}{2} \gamma^{\mu\nu}_{ab} Z_{\mu\nu} + \tfrac{1}{5!} \gamma^{\mu_1 \cdots \mu_5}_{ab} Z_{\mu_1 \cdots \mu_5} ,$$

where Q_a transforms in the spinor representation Δ of $\mathrm{Spin}(1,10)$. The right-hand side is simply the decomposition of $S^2\Delta = \Lambda^1 \oplus \Lambda^2 \oplus \Lambda^5$ in terms of irreducible representations of $\mathrm{Spin}(1,10)$. Under $\mathrm{Spin}(1,9)$, Δ decomposes as $\Delta = \Delta_+ \oplus \Delta_-$, where $\Delta_\pm$ are the chiral spinor representations of $\mathrm{Spin}(1,9)$. Let $Q^\pm_\alpha$ denote the corresponding generators. Write the Lie brackets $[Q^\pm_\alpha, Q^\pm_\beta]$ and $[Q^+_\alpha, Q^-_\beta]$ in terms of irreducible representations of $\mathrm{Spin}(1,9)$. The resulting superalgebra is the **IIA superalgebra**. Find examples of representations of this superalgebra corresponding to the following IIA branes: fundamental string, NS5, D0, D2 and D4, by identifying which charges (P_μ, $Z_{\mu\nu}$, $Z_{\mu_1\cdots\mu_5}$) must be turned on in the superalgebra, and writing down the corresponding spinor conditions.

Problem 6.4 Consider the **IIB superalgebra**

$$[Q^I_\alpha, Q^J_\beta] = \gamma^\mu_{\alpha\beta}\left(\delta^{IJ}P_\mu + \sigma_1^{IJ}Z_\mu + \sigma_3^{IJ}\widetilde{Z}_\mu\right) + \tfrac{1}{3!}\gamma^{\mu\nu\rho}_{\alpha\beta}\epsilon^{IJ}Z_{\mu\nu\rho}$$
$$+ \tfrac{1}{5!}\gamma^{\mu_1\cdots\mu_5}_{\alpha\beta}\left(\delta^{IJ}Z^+_{\mu_1\cdots\mu_5} + \sigma_1^{IJ}W^+_{\mu_1\cdots\mu_5} + \sigma_3^{IJ}\widetilde{W}^+_{\mu_1\cdots\mu_5}\right),$$

where $I,J = 1,2$ and $\mathfrak{g}_1 = 2\Delta_+$ consists of two copies of the positive-chirality spinor representation of $\mathrm{Spin}(1,9)$. Find examples of representations of this superalgebra corresponding to the following IIB branes: fundamental string, NS5, D(−1), D1 and D3, D5 and D7 by identifying which charges must be turned on in the superalgebra, and writing down the corresponding spinor conditions.

References

[1] K. Kamimura and M. Sakaguchi, "$osp(1|32)$ and extensions of super-$AdS_5 \times S^5$ algebra," *Nucl. Phys.* **B662** (2003) 491–510, hep-th/0301083.

[2] P. K. Townsend, "M-theory from its superalgebra," hep-th/9712004.