Annual Review Report

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Abstract

This report summarises the bulk of the work that has been undertaken since November 2021. In particular, semi-infinite cohomology of graded Lie algebras is first introduced in a general setting, after which considering the special case of the Virasoro algebra elucidates its relationship to two-dimensional meromorphic conformal field theory (2d CFT). At this point, the vertex operator algebra (VOA) construction is introduced. Using VOAs and spectral sequences, the spectrum of the original Gomis-Ooguri string, containing 24 transverse free bosons and a $\beta\gamma$-system of weight $(h_\beta, h_\gamma) = (1, 0)$, is reproduced by explicitly computing the BRST cohomology. Doing so also reveals that one has an infinite number of ″pictures″ labelled by positive integers giving rise to isomorphic BRST cohomologies. This arises from having an infinite number of choices for the $\beta\gamma$-vacuum, all of which sit in different representations of the $\beta\gamma$-algebra. Similar methods are then applied to the NSR string in this non-relativistic Gomis-Ooguri limit, which reveals a spectrum containing 8 free bosons and a $\beta\gamma$-system of the same weights that one would expect, along with 8 free Majorana fermions and a bc-system of weights $(h_b, h_c) = (\frac{1}{2}, \frac{1}{2})$. There are still an infinite number of pictures but they are now labelled by points in the upper half plane of $\mathbb{Z} \times \mathbb{Z}$ or $\mathbb{Z} \times (\mathbb{Z} + \frac{1}{2})$, depending on whether we are in the R or NS sectors respectively. Extra details of the algebraic frameworks introduced are provided in the various appendices.

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Chapter 1

Introduction

A theory of quantum gravity is a Holy Grail that has been long sought after by many modern theoretical physicists. Although such a theory describing our observable reality has eluded us for decades, the quest for quantum gravity still goes on. String theory is (or if it makes some people more comfortable, used to be) one of the most promising candidates for a theory quantum gravity. It gained massive amounts of popularity in the 1980s and 1990s due to profound results such as anomaly cancellation in superstring theory [1] and the existence of dualities relating the five known types of string theory [2, 3].

Probing nature at the most fundamental scale is rather strong motivation to invent new tools to do so. In other words, the rising popularity of string theory made a number of areas of pure mathematics research more active too. String theory not only made use of mathematical tools such as homological algebra, Lie algebras, representation theory and algebraic geometry, but also provided inspiration to these fields by being a useful toy model in many cases, from which more general structures could be built. A well-known example is the emergence of mirror symmetry as a new branch of mathematics in its own right after the work of Candelas, de la Ossa, Green and Parks [4] showed their usefulness in making predictions in enumerative geometry. Another example is the study of vertex operator algebras. A vertex operator algebra is the mathematical construction underlying two-dimensional meromorphic[§] conformal field theories (2d CFT) that arise in string theory. Using this as a source of inspiration [5, 6], Borcherds proved the monstrous moonshine conjecture [5].

Meanwhile, in 1984, Feigin introduced the construction of semi-infinite cohomology of Lie algebras [7]. Frenkel, Garland and Zuckerman (FGZ) then developed this construction in 1986 and applied them to bosonic string theory [8]. They demonstrated that what physicists studied as the BRST cohomology of physical states is nothing but the semi-infinite cohomology of the Virasoro algebra relative to its centre. This provides a mathematically solid and rigorous basis for studying string theory spectra. The semi-infinite cohomology of the Virasoro algebra is particularly interesting since it occurs as a vertex operator algebra, making it even easier to study in detail.

[§]In this report, we will only study meromorphic CFTs.

In 2000, Gomis and Ooguri constructed a theory of quantum strings on a background admitting Galilean invariance. Gomis-Ooguri type strings, non-relativistic strings with target space Galilean symmetry but worldsheet conformal symmetry, are susceptible to scrutiny using vertex operator algebra techniques. The behaviour of these types of strings is interesting to look at for various reasons, a key one being wanting to understand non-relativistic quantum gravity theories better. One can ask the question if a full-fledged relativistic quantum theory of gravity can be obtained as an extension of a non-relativistic one [9, 10], and non-relativistic string theory provides a way of probing this question. Naturally, this only opens more paths to explore, such as the AdS/CFT correspondence in such a setting [11] or non-relativistic analogues of D-branes [12].

In this report, various different areas of mathematics and physics are brought together. Techniques from semi-infinite cohomology, representation theory of infinite dimensional Lie algebras, vertex operator algebras and spectral sequences are all used in calculations in some context of non-relativistic string theory. The structure of this report is as follows. Chapter 2 provides an introduction to semi-infinite cohomology of graded infinite-dimensional Lie algebras, based on the paper [8] by FGZ. This chapter concludes with a segue to vertex operator algebras and 2d CFT by describing how the BRST cohomology emerges as a special case of semi-infinite cohomology. Chapter 3 then introduces the language of vertex operator algebras in detail and sets up the BRST formalism in this manner. These techniques are then applied to compute the BRST cohomology of the bosonic Gomis-Ooguri string using in Chapter 4, with the aid of spectral sequences. This reproduces the spectrum obtain in [13], with more care being given to the choice of vacuum. In Chapter 5, a Gomis-Ooguri version of the NSR string, which admits $N = 1$ superconformal invariance on the worldsheet, is constructed. Using analogous arguments as those in Chapter 4, the spectrum of this string is obtained by computation of the BRST cohomology. Chapter 6 concludes and briefly describes potential next steps.

Chapter 2

Semi-Infinite Cohomology of Graded Infinite-Dimensional Lie Algebras

This chapter will pedagogically introduce semi-infinite cohomology, in the context of graded ∞-dimensional Lie algebras. It is based primarily on the FGZ's paper [8].

2.1 Building the Space of Semi-Infinite Forms

Let $\mathfrak{g} = \bigoplus_{n\in\mathbb{Z}}\mathfrak{g}_n$ be a graded Lie algebra over $\mathbb{C}$, with $\dim \mathfrak{g}_n < \infty \; \forall n \in \mathbb{Z}$. Let $\mathfrak{g}_\pm := \bigoplus_{\pm n>0}\mathfrak{g}_n$. Let $\{e_i\}_{i\in\mathbb{Z}}$ be a basis for $\mathfrak{g}$ such that if $e_i \in \mathfrak{g}_n$ (for some $i, n \in \mathbb{Z}$), then either $e_{i+1} \in \mathfrak{g}_n$ or $e_{i+1} \in \mathfrak{g}_{n+1}$. Let $\mathfrak{g}' = \bigoplus_{n\in\mathbb{Z}}\mathfrak{g}'_n$ be the restricted dual of $\mathfrak{g}$ with $\mathfrak{g}'_n = \mathfrak{g}^*_n = \mathrm{Hom}(\mathfrak{g}_n, \mathbb{C})$. Let $\{e'_i\}_{i\in\mathbb{Z}}$ be the dual basis for $\mathfrak{g}'$ (so $\langle e'_i, e_j\rangle := e'_i(e_j) = e_j(e'_i) = \delta_{ij}$).

We may define a Clifford algebra $\mathrm{Cl}(\mathfrak{g} \bigoplus \mathfrak{g}')$ with respect to the bilinear pairing $\langle -, -\rangle : \mathfrak{g} \times \mathfrak{g}' \to \mathbb{C}$ as follows: As a vector space, $\mathrm{Cl}(\mathfrak{g} \bigoplus \mathfrak{g}') \cong \mathfrak{g} \bigoplus \mathfrak{g}'$. For any $x + x' \in \mathrm{Cl}(\mathfrak{g} \bigoplus \mathfrak{g}')$, we define the product "$\cdot$" of the algebra as

$$(x + x') \cdot (x + x') =: (x + x')^2 = \langle x, x'\rangle 1. \tag{2.1}$$

For a more general combination of elements,

$$(a + b') \cdot (c + d') + (c + d') \cdot (a + b') = \langle a, d'\rangle 1 + \langle c, b'\rangle 1 \tag{2.2}$$

so if $c + d' = a + b'$, we get

$$(a + b') \cdot (a + b') + (a + b') \cdot (a + b') = \langle a, b'\rangle 1 + \langle a, b'\rangle 1 \iff (a + b')^2 = \langle a, b'\rangle 1. \tag{2.3}$$

Definition 1: The **space of semi-infinite forms** $\Lambda^{\bullet}_{\infty}$ is the space spanned by monomials

$$\omega := e'_{i_1} \wedge e'_{i_2} \wedge \ldots \tag{2.4}$$

where $i_1 > i_2 > \ldots$ and $\exists\, N(\omega) \in \mathbb{Z}$ such that $i_{k+1} = i_k - 1 \; \forall k > N(\omega)$.

Definition 2: Let $x \in \mathfrak{g}$, $x' \in \mathfrak{g}'$. We define two endomorphisms of $\Lambda^{\bullet}_{\infty}$. The first is the **contraction** $\iota(x)\colon \Lambda^{\bullet}_{\infty} \to \Lambda^{\bullet}_{\infty}$, whose action on monomials is given by

$$\iota(x)e'_{i_1} \wedge e'_{i_2} \wedge \cdots = \sum_{k\geq 1}(-1)^{k-1}\langle x, e'_{i_k}\rangle e'_{i_1} \wedge e'_{i_2} \wedge \cdots \wedge \widehat{e'_{i_k}} \wedge \ldots, \tag{2.5}$$

where the hat denotes omission. The second is the **exterior product** $\varepsilon(x')\colon \Lambda^{\bullet}_{\infty} \to \Lambda^{\bullet}_{\infty}$ defined as

$$\varepsilon(x')e'_{i_1} \wedge e'_{i_2} \wedge \cdots = x' \wedge e'_{i_1} \wedge e'_{i_2} \wedge \ldots \tag{2.6}$$

Proposition 2.1: *For all $x, y \in \mathfrak{g}$ and $x', y' \in \mathfrak{g}'$, the following anticommutation relations hold:*

$$\begin{aligned} [\iota(x), \iota(y)]_+ = [\varepsilon(x'), \varepsilon(y')]_+ = 0 \\ [\iota(x), \varepsilon(x')]_+ = \langle x, x'\rangle. \end{aligned} \tag{2.7}$$

Proof: It suffices to prove these for monomials since any element in $\Lambda^{\bullet}_{\infty}$is just a $\mathbb{C}$-linear combination of these. The vanishing of the anticommutators of two contractions or two exterior products follows from the antisymmetry of wedge products of e'_i. To show the non-trivial anticommutator, we simply make use of the definitions of the endomorphisms

$$\begin{aligned} [\iota(x), \varepsilon(x')]_+ e'_{i_1} \wedge e'_{i_2} \wedge \ldots &= \langle x, x'\rangle e'_{i_1} \wedge e'_{i_2} \wedge \cdots + x' \wedge \sum_{k\geq 1}(-1)^{k}\langle x, e'_{i_k}\rangle e'_{i_1} \wedge \cdots \wedge \widehat{e'_{i_k}} \wedge \ldots \\ &\quad + x' \wedge \sum_{k\geq 1}(-1)^{k-1}\langle x, e'_{i_k}\rangle e'_{i_1} \wedge \cdots \wedge \widehat{e'_{i_k}} \wedge \ldots \\ &= \langle x, x'\rangle e'_{i_1} \wedge e'_{i_2} \wedge \ldots \end{aligned}$$

■

Proposition 2.1 implies that $\iota(x)^2 = \varepsilon(x')^2 = 0$ for all $x \in \mathfrak{g}$, $x' \in \mathfrak{g}'$.

Proposition 2.2: *$\lambda^{\bullet}_{\infty}$ admits a Clifford module structure over* $\mathrm{Cl}(\mathfrak{g} \bigoplus \mathfrak{g}')$.

Proof: Define $\kappa : \mathrm{Cl}(\mathfrak{g} \bigoplus \mathfrak{g}') \to \mathrm{End}(\Lambda^{\bullet}_{\infty})$ via $\kappa(x + x') := \iota(x) + \varepsilon(x')$. To prove the proposition we need to show that $(\kappa(x + x'))^2 = \langle x, x'\rangle$.

$$(\kappa(x + x'))^2 = \iota(x)^2 + \varepsilon(x')^2 + \iota(x)\varepsilon(x') + \varepsilon(x')\iota(x) = \langle x, x'\rangle.$$

■

2.2 Constructing a Representation

Definition 3: Just like with any other Lie algebra, we can define the **adjoint representation** of $\mathfrak{g}$ via a linear map $\mathrm{ad}\colon \mathfrak{g} \longrightarrow \mathrm{End}(\mathfrak{g})$

$$\mathrm{ad}_x := [x, -] \quad \forall x \in \mathfrak{g}. \tag{2.8}$$

Similarly, the **coadjoint representation** of $\mathfrak{g}$ is give by a linear map $\mathrm{ad}'\colon \mathfrak{g} \longrightarrow \mathrm{End}(\mathfrak{g}')$ such that $\forall x \in \mathfrak{g}$, $y' \in \mathfrak{g}'$,

$$\mathrm{ad}'_x(y') := -y' \circ \mathrm{ad}'_x = -y'([x, -]) \in \mathfrak{g}'. \tag{2.9}$$

We would now like to make $\Lambda^{\bullet}_{\infty}$ a representation of $\mathfrak{g}$. However (spoiler alert), unlike $\Lambda^{*}\mathfrak{g}' := \bigoplus_{n\in\mathbb{N}}\Lambda^{n}\mathfrak{g}'$, $\Lambda^{\bullet}_{\infty}$ is at best a representation of some 1-dimensional central extension $\hat{g}$ rather than one of $\mathfrak{g}$ itself [8]. Nonetheless, let us start with the most natural thing, which would be a linear map $\rho\colon \mathfrak{g} \to \mathrm{End}(\Lambda^{\bullet}_{\infty})$ that extends the coadjoint action to to the monomials that span the semi-infinite forms

$$\rho(x)e'_{i_1} \wedge e'_{i_2} \wedge \cdots = \sum_{k\geq 1} e'_{i_1} \wedge e'_{i_2} \wedge \cdots \wedge \mathrm{ad}'_x\, e'_{i_k} \wedge \cdots = \sum_{k\geq 1} \varepsilon(\mathrm{ad}'_x\, e'_{i_k})\iota(e_{i_k})e'_{i_1} \wedge e'_{i_2} \wedge \dots \tag{2.10}$$

Proposition 2.3: *The following commutation relations hold for all $x, y \in \mathfrak{g}$, $y' \in \mathfrak{g}'$:*

$$[\rho(x), \iota(y)] = \iota(\mathrm{ad}_x\, y) \qquad [\rho(x), \varepsilon(y')] = \varepsilon(\mathrm{ad}'_x\, y'). \tag{2.11}$$

The summation in (2.10) is only finite when $x \notin \mathfrak{g}_0$. This is because for $x \in \mathfrak{g}_n$ and $e'_{i_k} \in \mathfrak{g}'_{m_k}$ $\mathrm{ad}'_x\, e'_{i_k} := -e'_{i_k}([x, -])$ needs to take in an element of $\mathfrak{g}_{m_k-n}$ to be non-zero by definition of a grading-comaptible Lie bracket and dual basis pairing. Thus, $\mathrm{ad}'_x\, e'_{i_k} \in \mathfrak{g}'_{m_k-n}$. Since any semi-infinite form monomial ω is "filled" once $k = N(\omega)$, when $n \neq 0$, we will eventually reach some value of k for which $m_k - n = m_{\bar{k}}$, where $\bar{k} > N(\omega)$ and not equal to k. But because $\bar{k} > N(\omega)$, every dual basis element living in $\mathfrak{g}_{m_{\bar{k}}}$ will be present in the monomial ω. Thus, we would inevitably get a contraction of two identical dual basis elements, which is equal to zero by (2.7). Hence, for $n \neq 0$, the sum (2.10) will truncate to a finite one. However, this duplication of dual basis elements in the monomial will no longer happen when $x \in \mathfrak{g}_0$ since $\mathrm{ad}'_x\colon \mathfrak{g}'_{m_k} \to \mathfrak{g}'_{m_k}$.

For a sensible definition of $\rho\colon \mathfrak{g}_0 \to \mathrm{End}(\Lambda^{\bullet}_{\infty})$, we start by defining a vacuum semi-infinite form ω_0 that obeys $\rho([x, y])\omega_0 = \lambda(x, y)\omega_0$ for all $x \in \mathfrak{g}_n$, $y \in \mathfrak{g}_{-n}$ where $n \in \mathbb{Z} \setminus \{0\}$ and $\lambda \in \Lambda^2(\mathfrak{g}')$. The standard way to construct such a vacuum is by choosing i_0 such that $e'_{i_0} \in \mathfrak{g}_m \implies e'_{i_0+1} \in \mathfrak{g}_{m+1}$ and then letting

$$\omega_0 := e'_{i_0} \wedge e'_{i_0-1} \wedge e'_{i_0-2} \wedge \dots \tag{2.12}$$

Hence, ω_0 is the ordered wedge product of the dual basis elements spanning $\bigoplus_{n\leq i_0}\mathfrak{g}'_n$. Then for a given ω_0, choose a $\beta \in \mathfrak{g}'_0$ such that $\beta([\mathfrak{g}_0, \mathfrak{g}_0]) = 0$, and define $\rho(x)\omega_0 = \langle\beta, x\rangle\omega_0$. By demanding that the anticommutation relations (2.11) hold, we may uniquely extend such an action of ρ to all of $\mathfrak{g}$. Explicitly:

$$\rho(x) := \sum_{i\in\mathbb{Z}} :\varepsilon(\mathrm{ad}'_x\, e'_i)\iota(e_i): + \langle\beta, x\rangle, \tag{2.13}$$

where we have defined the **normal-ordered product** with respect to the vacuum ω_0 as

$$:\varepsilon(\mathrm{ad}'_x\, e'_i)\iota(e_i): := \begin{cases} \varepsilon(\mathrm{ad}'_x\, e'_i)\iota(e_i), & i > i_0 \\ -\iota(e_i)\varepsilon(\mathrm{ad}'_x\, e'_i), & i \leq i_0 \end{cases}. \tag{2.14}$$

Notice that this summation really just looks like an extension of what we wrote on the RHS of equation (2.10) and when $x \notin \mathfrak{g}_0$, we obtain our original definition of ρ. It is worth mentioning that we may equivalently write this normal-ordered sum [14] as

$$\rho(x) = \sum_{i\in\mathbb{Z}} :\iota(\mathrm{ad}_x\, e_i)\varepsilon(e'_i): + \langle\beta, x\rangle. \tag{2.15}$$

This also satisfies (2.11). We may intuitively understand the equivalence between the two sums as follows. In the original definition (2.13), each normal-ordered term in the sum involves a contraction with a dual basis element in $\mathfrak{g}_{m_i}$ and an exterior product with an element in $\mathfrak{g}'_{m_i-n}$, where $x \in \mathfrak{g}_n$. Consequently, the sum over $i \in \mathbb{Z}$ is a sum over basis elements of $\mathfrak{g}'_{m_i-n}$ and $\mathfrak{g}_{m_i}$. Now in (2.15), each term involves an exterior product with a dual basis element of $\mathfrak{g}'_{m_i}$ and a contraction with an element in $\mathfrak{g}_{m_i+n}$. Hence, the sum over $i \in \mathbb{Z}$ is in some sense "shifted" by n; we now sum over basis elements of $\mathfrak{g}'_{m_i}$ and $\mathfrak{g}_{m_i+n}$ instead. The action on monomials is still the same too. Both definitions add and remove elements belonging to the same graded subspaces of $\mathfrak{g}'$.

Let us take $x \in \mathfrak{g}_n$ and $y \in \mathfrak{g}_{-n}$ for $n \neq 0$. Then $\rho([x,y])\omega_0 = \langle \beta, [x,y]\rangle \omega_0 = -d_{\mathrm{LA}}\beta(x,y)\omega_0$, where d_{LA} is the differential in Lie algebra cohomology. Comparing to our original requirement for a vacuum semi-infinite for, we learn that $\lambda = -d_{\mathrm{LA}}\beta$ is actually coboundary, which means it belongs to the trivial class in $H^2(\mathfrak{g})$. Recalling that

$$H^2(\mathfrak{g}) \cong \{\text{equivalence classes of central extension of } \mathfrak{g}\},$$

we observe that there is a link (albeit a flimsy one) between the space of semi-infinite forms and central extensions of $\mathfrak{g}$. The following propositions shed more light on this link and will prove our claim that $\Lambda^{\bullet}_{\infty}$ is a $\hat{g}$-module.

Proposition 2.4: *There exists a two-cocycle $\gamma \in H^2(\mathfrak{g})$ depending on the choice of vacuum ω_0 and β such that*

1. $\gamma(\mathfrak{g}_m, \mathfrak{g}_n) = 0 \; \forall m + n \neq 0$
2. $[\rho(x), \rho(y)] = \rho([x,y]) + \gamma(x,y)$.

Proposition 2.5: *If γ is a coboundary, then there exists a choice of β for a given ω_0 such that $\gamma = 0 \in \Lambda^2(\mathfrak{g})$.*

Proof: If γ is a coboundary, there exists $\alpha \in \mathfrak{g}'$ such that $\gamma = d_{\mathrm{LA}}\alpha$ and recall that $\gamma(x,y) = d_{\mathrm{LA}}\alpha(x,y) = -\alpha([x,y])$. In particular, this means that $\alpha \in \mathfrak{g}_0$. We currently have a $\rho\colon \mathfrak{g} \to \mathrm{End}(\Lambda^{\bullet}_{\infty})$ that obeys2.4. Let us define a new representation $\tilde{\rho}\colon \mathfrak{g} \to \mathrm{End}(\Lambda^{\bullet}_{\infty})$ given by $\tilde{\rho}(x) := \rho(x) + \langle \alpha, x\rangle$. Then

$$\begin{aligned}
[\tilde{\rho}(x), \tilde{\rho}(y)] &:= [\rho(x) + \langle \alpha, x\rangle, \rho(y) + \langle \alpha, y\rangle] \\
&= [\rho(x), \rho(y)] \\
&= \rho([x,y]) + \gamma(x,y) && \text{(by proposition 2.4)} \\
&= \rho([x,y]) + d_{\mathrm{LA}}\alpha(x,y) && \text{(by definition of } \gamma) \\
&= \rho([x,y]) - \langle \alpha, [x,y]\rangle \\
&= \tilde{\rho}([x,y]). && \text{(by definition of } \tilde{\rho})
\end{aligned}$$

Any ω_0 defines $\rho\colon \mathfrak{g} \to \Lambda^{\bullet}_{\infty}$ satisfying proposition 2.4 using some choice of β. If $\exists \alpha \in \mathfrak{g}_0$ such that $\gamma = d_{\mathrm{LA}}\alpha$, then one can make the modification $\beta \to \tilde{\beta} := \beta + \alpha$ so that $\gamma = 0$.

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Corollary 2.5.1: *The space of semi-infinite forms $\Lambda^{\bullet}_{\infty}$ is, in general, not $\mathfrak{g}$-module, but rather a $\hat{\mathfrak{g}}$-module, where $\hat{g}$ is a central extension of $\mathfrak{g}$.*

From this point on, we will assume that by passing to a central extension of $\mathfrak{g}$ (if needed) $\gamma = 0$ and thus $\rho\colon \mathfrak{g} \to \Lambda^{\bullet}_{\infty}$ is a genuine representation (so $[\rho(x), \rho(y)] = \rho([x, y])$.

2.2.1 Gradings

There exist two natural gradings one can define on $\Lambda^{\bullet}_{\infty}$.

Definition 4: $\forall x \in \mathfrak{g}$, $x' \in \mathfrak{g}'$,

$$\operatorname{Deg} \iota(x) = -1 \qquad \operatorname{Deg} \varepsilon(x') = 1. \tag{2.16}$$

Fixing $\operatorname{Deg} \omega_0 \in \mathbb{Z}$, this defines the grading **Deg** on $\Lambda^{\bullet}_{\infty}$. We will sometimes refer to this grading as the *ghost number*, the name being motivated by BRST quantisation in physics.

Since $\operatorname{Deg} \rho = 0$, this makes $\Lambda^{m}_{\infty} := \{\omega \in \Lambda^{\bullet}_{\infty} \mid \operatorname{Deg} \omega = m\}$ a $\mathfrak{g}$-module $\forall m \in \mathbb{Z}$.

Definition 5: $\forall x \in \mathfrak{g}_n$, $x' \in \mathfrak{g}'_n$,

$$\deg \iota(x) = n \qquad \deg \varepsilon(x') = -n. \tag{2.17}$$

Fixing $\deg \omega_0 \in \mathbb{Z}$, this defines the grading **deg** on $\Lambda^{\bullet}_{\infty}$.

Let $\Lambda^{m;n}_{\infty} := \{\omega \in \Lambda^{m}_{\infty} \mid \deg \omega = n\}$ and $\Lambda^{\bullet;n}_{\infty} := \{\omega \in \Lambda^{\bullet}_{\infty} \mid \deg \omega = n\}$. For all $x \in \mathfrak{g}_k$, $\rho(x)\colon \Lambda^{m;n}_{\infty} \to \Lambda^{m;n+k}_{\infty}$. Hence, deg makes Λ^{m}_{∞} and $\Lambda^{\bullet}_{\infty}$ graded $\mathfrak{g}$-modules.

Definition 6: The **category $\mathcal{O}_0$** comprises of graded $\mathfrak{g}$-modules $\mathfrak{M} = \bigoplus_{n\in\mathbb{Z}} \mathfrak{M}_n$ such that $\dim \mathfrak{M}_n < \infty$ and for all $n > n_0$, $\dim \mathfrak{M}_n = 0$, for some $n_0 \in \mathbb{Z}$.

Regardless of how $\deg \omega_0$ is fixed, the structure of $\Lambda^{\bullet}_{\infty}$ and the construction of deg is such that $\dim \Lambda^{*;n}_{\infty} < \infty$ and is zero for all $n > n_0$ for some $n_0 \in \mathbb{Z}$. Hence, $\Lambda^{*;n}_{\infty} \in \mathcal{O}_0$.

Definition 7: The **category $\mathcal{O} \supset \mathcal{O}_0$** comprises of graded $\mathfrak{g}$-modules $\mathfrak{M} = \bigoplus_{n\in\mathbb{Z}} \mathfrak{M}_n$ such that $\dim \mathfrak{M}_n < \infty$ and the $\mathfrak{g}_+$-submodule $\{\mathcal{U}(\mathfrak{g}_+)v \mid v \in \mathfrak{M}\}$, where $\mathcal{U}(\mathfrak{g}_+)$ denotes the universal enveloping algebra of $\mathfrak{g}_+$, is finite dimensional for any $v \in \mathfrak{M}$.

2.3 Chain Complex Structure

Consider an arbitrary graded $\mathfrak{g}$-module $\mathfrak{M} \in \mathcal{O}_0$ with representation $\pi\colon \mathfrak{g} \to \operatorname{End} \mathfrak{M}$. Let $\deg v = n$ for all $v \in \mathfrak{M}_n$. Defining $\deg(v \otimes \omega) := \deg v + \deg \omega$ turns $\mathfrak{M} \bigotimes \Lambda^{\bullet}_{\infty}$ into a graded space, with each graded subspace being finite dimensional. Then $\theta\colon \mathfrak{g} \to \operatorname{End}(\mathfrak{M} \bigotimes \Lambda^{\bullet}_{\infty})$ given by $\theta(x) = \pi(x) + \rho(x)$ makes $\mathfrak{M} \bigotimes \Lambda^{\bullet}_{\infty}$ an object in category $\mathcal{O}_0$.

Definition 8: The **differential** d is given by

$$d := \sum_{i\in\mathbb{Z}} \pi(e_i)\varepsilon(e_i') + \sum_{i<j} :\iota([e_i, e_j])\varepsilon(e_j')\varepsilon(e_i'): . \tag{2.18}$$

Proposition 2.6: $d^2 = 0$.

The proof (see Appendix B) makes use of a result by Akman [15] to make it far less tedious than a brute-force approach. A key feature of this proof is that the statement of proposition 2.6 is equivalent to

$$\theta(x) = [d, \iota(x)]_+. \tag{2.19}$$

Definition 9: $\{\mathfrak{M} \otimes \Lambda_\infty^\bullet, d\}$ is a (graded) chain complex

$$\ldots \xrightarrow{d} \mathfrak{M} \otimes \Lambda_\infty^{m-1} \xrightarrow{d} \mathfrak{M} \otimes \Lambda_\infty^{m} \xrightarrow{d} \mathfrak{M} \otimes \Lambda_\infty^{m+1} \xrightarrow{d} \ldots$$

and the corresponding cohomology $H_\infty^\bullet(\mathfrak{g};\mathfrak{M})$ is known as the **semi-infinite cohomology of $\mathfrak{g}$** with values in $\mathfrak{M}$. Explicitly,

$$H_\infty^m(\mathfrak{g};\mathfrak{M}) = \frac{\ker\left(d\colon \mathfrak{M}\otimes\Lambda_\infty^m \to \mathfrak{M}\otimes\Lambda_\infty^{m+1}\right)}{\operatorname{im}\left(d\colon \mathfrak{M}\otimes\Lambda_\infty^{m-1} \to \mathfrak{M}\otimes\Lambda_\infty^{m}\right)}. \tag{2.20}$$

The differential raises Deg by 1 and leaves deg unchanged, so one can consider the chain complex for each deg too

$$\ldots \xrightarrow{d} \left(\mathfrak{M} \otimes \Lambda_\infty^{m-1}\right)^n \xrightarrow{d} \left(\mathfrak{M} \otimes \Lambda_\infty^{m}\right)^n \xrightarrow{d} \left(\mathfrak{M} \otimes \Lambda_\infty^{m+1}\right)^n \xrightarrow{d} \ldots$$

Then $H_\infty^m(\mathfrak{g};\mathfrak{M}) = \bigoplus_{n\in\mathbb{Z}} H_\infty^{m;n}(\mathfrak{g};\mathfrak{M})$, where

$$H_\infty^{m;n}(\mathfrak{g};\mathfrak{M}) = \frac{\ker\left(d\colon \left(\mathfrak{M}\otimes\Lambda_\infty^m\right)^n \to \left(\mathfrak{M}\otimes\Lambda_\infty^{m+1}\right)^n\right)}{\operatorname{im}\left(d\colon \left(\mathfrak{M}\otimes\Lambda_\infty^{m-1}\right)^n \to \left(\mathfrak{M}\otimes\Lambda_\infty^{m}\right)^n\right)}. \tag{2.21}$$

2.4 A Hermitian Form

Let $\mathfrak{g}$ admit an involutive anti-linear automorphism $\sigma : \mathfrak{g} \to \mathfrak{g}$ such that $\sigma : \mathfrak{g}_n \to \mathfrak{g}_{-n}$. This induces an involutive anti-linear automorphism on $\mathfrak{g}'$ (which we will also call σ) via

$$\langle x, \sigma(y')\rangle := \overline{\langle \sigma(x), y'\rangle}. \tag{2.22}$$

The bar denotes complex conjugation. Then

$$\sigma^2 = \mathrm{id}_{\mathfrak{g}} \implies \langle x, y'\rangle = \langle x, \sigma^2(y')\rangle = \overline{\langle \sigma(x), \sigma(y')\rangle}.$$

We now make use of σ to construct a Hermitian form $\{-,-\}\colon \mathfrak{g}\times\mathfrak{g} \to \mathbb{C}$ such that

$$\varepsilon(x')^\dagger = -\varepsilon(\sigma(x')) \qquad \iota(x)^\dagger = -\iota(\sigma(x)). \tag{2.23}$$

Proposition 2.7:

$$\rho(x)^\dagger = -\rho(\sigma(x)) \qquad \sigma(\beta) = -\beta. \tag{2.24}$$

To make $\{-,-\}$ unique, we choose two monomials forms $\omega_1 = e'_{i_1} \wedge e'_{i_2} \wedge \ldots$ and $\omega_2 = e'_{j_1} \wedge e'_{j_2} \wedge \ldots$ such that $\{e'_{i_k}\}_{k\geq 1} \bigcup \{\sigma(e'_{j_k})\}_{k\geq 1}$ is a basis for $\mathfrak{g}'$, in which case

$$\cdots \wedge \sigma(e'_{j_2}) \wedge \sigma(e'_{j_1}) \wedge e'_{i_1} \wedge e'_{i_2} \wedge \ldots$$

is known as the volume element. Since any such ω_1 and ω_2 can only differ by some multiple, we rescale $\{-,-\}$ so that it becomes Hermitian.

We now assume $\mathfrak{M}$ also has a non-degenerate Hermitian form such that

$$\pi(x)^\dagger = -\pi(\sigma(x)). \tag{2.25}$$

Then define a Hermitian form on $\mathfrak{M} \bigotimes \Lambda^\bullet_\infty$ as the tensor product of the respective Hermitian forms.

Proposition 2.8:

$$d^\dagger = d. \tag{2.26}$$

A Hermitian form on $\mathfrak{M} \bigotimes \Lambda^\bullet_\infty$ allows us to write down a *Poincaré Duality* theorem:

Theorem 2.9: *The anti-dual of $H^{m;n}_\infty(\mathfrak{g};\mathfrak{M})$ is isomorphic to $H^{-m;n}_\infty(\mathfrak{g};\mathfrak{M})$.*

2.5 The Relative Subcomplex

Let $\mathfrak{h} \subset \mathfrak{g}_0$ be a subalgebra. We define a subspace

$$C^\bullet_\infty(\mathfrak{g},\mathfrak{h};\mathfrak{M}) := \{w \in \mathfrak{M} \bigotimes \Lambda^\bullet_\infty \mid \iota(x)w = \theta(x)w = 0 \quad \forall x \in \mathfrak{h}\}. \tag{2.27}$$

Equation (2.19) implies

$$\theta(x)w = 0 \iff \big(d\iota(x) + \iota(x)d\big)w = \iota(x)dw = 0 \;\; \forall w \in C^\bullet_\infty(\mathfrak{g},\mathfrak{h};\mathfrak{M}).$$

Consequently, for any $w \in C^\bullet_\infty(\mathfrak{g},\mathfrak{h};\mathfrak{M})$, $\iota(x)dw = 0$ and $\theta(x)dw = \big(d\iota(x) + \iota(x)d\big)dw = 0$, so $d\big(C^\bullet_\infty(\mathfrak{g},\mathfrak{h};\mathfrak{M})\big) \subseteq C^\bullet_\infty(\mathfrak{g},\mathfrak{h};\mathfrak{M})$.

Definition 10: Let

$$C^m_\infty(\mathfrak{g},\mathfrak{h};\mathfrak{M}) := \{w \in \mathfrak{M} \bigotimes \Lambda^m_\infty \mid \iota(x)w = \theta(x)w = 0 \quad \forall x \in \mathfrak{h}\}.$$

The **subcomplex relative to $\mathfrak{h}$** is the chain complex $\{C^\bullet_\infty(\mathfrak{g},\mathfrak{h};\mathfrak{M}), d\}$

$$\ldots \xrightarrow{d} C^{m-1}_\infty(\mathfrak{g},\mathfrak{h};\mathfrak{M}) \xrightarrow{d} C^m_\infty(\mathfrak{g},\mathfrak{h};\mathfrak{M}) \xrightarrow{d} C^{m+1}_\infty(\mathfrak{g},\mathfrak{h};\mathfrak{M}) \xrightarrow{d} \ldots$$

The cohomology of this relative subcomplex is denoted $H^\bullet(\mathfrak{g}, \mathfrak{h}; \mathfrak{M})$. An analogue of theorem 2.9 holds when $\mathfrak{M}$ is Hermitian. We now focus on the case where $\mathfrak{h} = \mathfrak{g}_0$.

Let $C^m := C^m(\mathfrak{g}, \mathfrak{g}_0; \mathfrak{M})$. There exists a natural bigraded structure $C^m = \bigoplus_{c-b=m} C^{b,c}$, where $C^{b,c}$ is spanned by monomials $e'_{i_1} \wedge e'_{i_2} \wedge \ldots$ with b being the number of $e'_i \in \mathfrak{g}'_-$ missing from the monomials and c being the number of $e'_i \in \mathfrak{g}'_+$ present in the monomoials. Thus, from definition 4, $c - b$ is the ghost number of elements in $C^{b,c}$. There also exists a canonical splitting $d = d_b + d_c$, where

$$d_b : C^{b,c} \longrightarrow C^{b-1,c} \qquad d_c : C^{b,c} \longrightarrow C^{b,c+1},$$

allowing $C^{b,c}$ to admit structures that look like those in Kahler geometry [8], but we will not probe them any further in this report.

Theorem 2.10: *When $\mathfrak{M} \in \mathcal{O}_0$ is a $\mathcal{U}(\mathfrak{g}_-)$-free module,*

$$H^m(\mathfrak{g}, \mathfrak{g}_0; \mathfrak{M}) = 0 \qquad \forall m \neq 0. \tag{2.28}$$

Theorem 2.10 is known as a vanishing theorem. Although we will not need to make use of this theorem when computing cohomologies in the context of Gomis-Ooguri strings, it is still a powerful result about the structure of semi-infinite cohomology that is worth mentioning. In relativistic string theory, $\mathfrak{g}$ is the Virasoro algebra and one possible use of this theorem arises when $\mathfrak{M}$ is the Fock module. This vanishing theorem can be proved using spectral sequences (see Appendix A for a review). A proof is outlined in [8].

2.6 Setup for String Theory Applications

The symmetry algebra $\mathfrak{g} = \bigoplus_{n\in\mathbb{Z}} \mathfrak{g}_n$ that one would consider for string theory is the Virasoro algebra, the unique central extension of the Witt algebra, which has

$$\mathfrak{g}_n = \mathbb{C}L_n \text{ for } n \neq 0, \quad \mathfrak{g}_0 = \mathbb{C}L_0 \bigoplus \mathbb{C}c,$$

and the generators $\{L_n, c\}_{n\in\mathbb{Z}}$ satisfy the following Lie algebra

$$[L_m, L_n] = (m-n)L_{m+n} + \frac{c}{12}m(m^2-1)\delta_{m+n.0}, \quad [L_n, c] = 0. \tag{2.29}$$

2.6.1 Relationship to 2d CFT

We choose the vacuum $\omega_0 = L'_1 \wedge L'_0 \wedge L'_{-1} \wedge \ldots$ and introduce generating functions, otherwise known as quantum fields

$$b(z) = \sum_{n\in\mathbb{Z}} b_n z^{-n-2} \qquad c(z) = \sum_{n\in\mathbb{Z}} c_n z^{-n+1} \tag{2.30}$$

$$T^{\mathfrak{M}}(z) = \sum_{n\in\mathbb{Z}} \pi(L_n) z^{-n-2} \qquad T^{bc}(z) = \sum_{n\in\mathbb{Z}} \rho(L_n) z^{-n-2} \tag{2.31}$$

where $b_n := \iota(L_n)$ and $c_n := \varepsilon(L'_{-n})$.

Proposition 2.11:

$$\rho(c) = \langle \beta, c\rangle \operatorname{Id}_{\Lambda^{\bullet}_{\infty}} = -26 \operatorname{Id}_{\Lambda^{\bullet}_{\infty}} \qquad \rho(L_m) = \sum_n (m-n) : c_{-n} b_{m+n} : . \tag{2.32}$$

The non-trivial anti-commutator in (2.7) now reads

$$c_n b_m + b_m c_n = \delta_{m+n,0}. \tag{2.33}$$

Thus, b_n and c_n are the modes of the BRST antighost and ghost fields respectively, which obey the mode algebra of a fermionic bc-system (see chapter 3). $T^{\mathfrak{M}}(z)$ and $T^{bc}(z)$ are the energy-momentum tensors of the CFTs of the matter fields that make up $\mathfrak{M}$ (e.g: 26 free bosons when $\mathfrak{M}$ is the Fock module) and the ghosts respectively.

Proposition 2.12: *T^{bc} can be written in terms of $b(z)$ and $c(z)$ as*

$$T^{bc} =: -2b(z)\left(\frac{d}{dz}c(z)\right) - \left(\frac{d}{dz}b(z)\right)c(z) : \tag{2.34}$$

We may define the BRST current

$$j(z) =: c(z)\big(T^{\mathfrak{M}}(z) + \frac{1}{2}T^{bc}(z)\big) : \tag{2.35}$$

and thus express the differential as

$$d = \oint_{C_z} \frac{dz}{2\pi i} j(z). \tag{2.36}$$

(2.36) is precisely the form of the BRST charge in 2d CFT.

To compute string spectra, we make use of the subcomplex of $\{\mathfrak{M} \bigotimes \Lambda^{\bullet}_{\infty}, d\}$ relative to the centre, $\mathfrak{z}$. In this case, since $\mathfrak{z} = \mathbb{C}c$ acts on $\mathfrak{M}$ by scalars, $H^m(\mathfrak{g}, \mathfrak{z}; \mathfrak{M})$ is non-trivial only when $\pi(z) = -\langle \beta, z\rangle \operatorname{Id}_{\mathfrak{M}}$ [8]. Hence, $\mathfrak{M}$ must be such that $\pi(c) = 26 \operatorname{Id}_{\mathfrak{M}}$. Consequently,

$$\theta(c)(v \otimes \omega) = (\pi(c) + \rho(c))(v \otimes \omega) = (26v) \otimes \omega + v \otimes (-26\omega) = 0.$$

This is the requirement that the matter fields in a string theory obey a CFT with central charge 26 to give rise to a CFT with vanishing central charge after coupling the matter fields to ghosts.

The fields that have been briefly introduced in this subsection are objects that appear in 2d CFTs, which have OPEs that are meromorphic functions on the Riemann sphere $\mathbb{C} \cup \{\infty\}$. These can be studied algebraically using vertex operator algebras.

Chapter 3

Vertex Operator Algebras in 2d CFTs

In this chapter, a vertex operator algebra (VOA) is constructed via set of axioms. These axioms encode the key features of meromorphic 2d CFTs. I Some algebraic notation is introduced and many useful properties are stated. Anticipating the application to string theory, the CFT of bc-systems is studied in general using the VOA language.

It should be noted that the terminology varies greatly in the literature. For example, in some works such as those by Akman [16] and Meurman and Primc [17], VOAs refer to very similar constructions as the one presented in this report while in some other works, such as this paper by Thielemans [18], the term "operator product algebra" is used instead. Such caveats should be kept in mind when consulting other sources.

3.1 Axioms

Definition 11: A **vertex operator algebra** is given by the following construction:

- A $\mathbb{Z}_2$-graded vector space $\mathfrak{M} = \mathfrak{M}_{\bar{0}} \oplus \mathfrak{M}_{\bar{1}}$ over $\mathbb{C}$ spanned by elements known as **states**.
- A homogeneous element $A \in \mathfrak{M}$ has *parity* $|A| = 0$ if $A \in \mathfrak{M}_{\bar{0}}$ or $|A| = 1$ if $A \in \mathfrak{M}_{\bar{1}}$. We will call such states *bosonic* and *fermionic* respectively.
- There exists a *vacuum state* $\Omega \in \mathfrak{M}_{\bar{0}}$.
- For every $A \in \mathfrak{M}$, there exists a field $A(z) \in \operatorname{End}\mathfrak{M}[[z, z^{-1}]]$ such that

$$\lim_{z \to 0} A(z)\Omega = A. \tag{3.1}$$

  This is known as the *state-field correspondence*.
- We will require that $\mathfrak{M} = \bigoplus_{n\in\mathbb{Z}} \mathfrak{M}^n$ belongs to the category $\mathcal{O}_0$, with $\pi \colon \mathfrak{g} \to \operatorname{End}\mathfrak{M}$ a Virasoro algebra representation that is graded by the eigenvalue of L_0, meaning

$$\pi(L_0)A = nA \qquad \forall A \in \mathfrak{M}^n. \tag{3.2}$$

  For reasons that will soon become clearer, we will simply write $L_n A$ for the action of the Virasoro algebra on $\mathfrak{M}$.

- There are set of bilinear brackets $[-,-]_n : \mathfrak{M} \otimes \mathfrak{M} \to \mathfrak{M}$, labelled by $n \in \mathbb{Z}$, defined by the **operator product expansion (OPE)**:

$$A(z)B(w) = \sum_{n \ll \infty} \frac{[A,B]_n(w)}{(z-w)^n}, \tag{3.3}$$

where the summation index $n \ll \infty$ indicates that there are only a finite number of singular terms (those with $n > 0$) in the sum. For our CFT purposes, OPEs are meant to be understood as radially ordered correlators. In other words, $|z| > |w|$ in the above and more generally, for $|z| > |w| > \cdots > |u|$,

$$\langle A(z)B(w)\dots C(u)\rangle = \sum_{n \ll \infty} \frac{1}{(z-w)^n} \langle [A,B]_n(w)\dots C(u)\rangle.$$

From (3.3), the brackets may be computed via Cauchy's Residue Theorem

$$[A,B]_n(w) = \oint_{C_w} \frac{dz}{2\pi i}(z-w)^{n-1}A(z)B(w), \tag{3.4}$$

where C_w is a positively oriented contour enclosing w. The OPEs obey the following supercommutator

$$A(z)B(w) - (-1)^{|A||B|}B(w)A(z) = 0. \tag{3.5}$$

The $(-1)^{|A||B|}$ is known as the *Koszul sign*.

- The *derivation* $\partial : \mathfrak{M} \to \mathfrak{M}$ is defined as $(\partial A)(z) := \frac{d}{dz}A(z)$.

- There exists a **Virasoro element** $T \in \mathfrak{M}_{\bar{0}}$ which admits the following OPE

$$T(z)T(w) = \frac{\frac{c}{2}\mathbb{1}(w)}{(z-w)^4} + \frac{2T(w)}{(z-w)^2} + \frac{\partial T(w)}{z-w} + \text{reg.} \tag{3.6}$$

where $\mathbb{1}(z) = \mathrm{id}_{\mathfrak{M}}$ and c is the **central charge** of T.

- A vector $A \in \mathfrak{M}$ has **conformal weight** h_A if $[T,A]_2 = h_A A$. For every such A, its corresponding field admits a mode expansion $A(z) = \sum_n A_n z^{-n-h_A}$, where $A_n \in \mathrm{End}\mathfrak{M}$. (3.6) implies that the Virasoro element has conformal weight 2, and thus a mode expansion $T(z) = \sum_n L_n z^{-n-2}$, where we regard the Virasoro generators $L_n \in \mathrm{End}\,\mathfrak{M}$ as a shorthand for $\pi(L_n) \in \mathrm{End}\,\mathfrak{M}$. The requirement of $\mathfrak{M}$ being graded by the eigenvalue of L_0 is equivalent to the requirement of $\mathfrak{M}$ admitting a basis consisting of vectors with definite conformal weights (see proposition 3.5).

- Without loss of generality, we will assume that $\forall A \in \mathfrak{M}$, $[T,A]_1 = \partial A$.

- If a vector $\phi \in \mathfrak{M}$ satisfies

$$[T,\phi]_2 = h\phi, \quad [T,\phi]_1 = \partial\phi, \quad [T,\phi]_{n>2} = 0, \tag{3.7}$$

then ϕ is a **conformal primary of weight** h.

It should be noted that this VOA is equipped with a Virasoro action and a graded structure that assumes L_0 is diagonalisable. These are restrictive assumptions and there exist logarithmic CFTs for which L_0 is not diagonalisable (see [19, 20]). More generally, VOAs, such as those in [21], do not need a Virasoro element at all.

3.2 Properties

We start by exploring some elementary properties of ∂.

Proposition 3.1: *The derivation ∂ satisfies the following properties:*

(a) $[\partial A, B]_n = -(n-1)[A, B]_{n-1}$. *Consequently,* $[\partial A, B]_1 = 0\ \forall A, B \in \mathfrak{M}$.

(b) $[A, \partial B]_n = (n-1)[A, B]_{n-1} + \partial[A, B]_n$.

(c) $\partial[A, B]_n = [\partial A, B]_n + [A, \partial B]_n$.

(d) $[T, A]_2 = h_A A \implies [T, \partial A]_2 = (h_A + 1)A$.

(e) $(\partial A)_n = -(n + h_A)A_n$, *where* $A(z) = \sum_n A_n z^{-n-h_A}$.

Proposition 3.1(a) implies that $[A, B]_{n\geq 0}$ determines $[A, B]_{n<0}$, the regular part of the OPE $A(z)B(w)$.

Proposition 3.2: *The brackets $[-,-]_n$ obey the following skew-symmetry condition:*

$$[A, B]_n - (-1)^{|A||B|+n}[B, A]_n = (-1)^{|A||B|+n} \sum_{l\geq 1} \frac{(-1)^l}{l!} \partial^l [B, A]_{l+n} = \sum_{l\geq 1} \frac{(-1)^{l+1}}{l!} \partial^l [A, B]_{l+n}. \tag{3.8}$$

This follows from (3.5) (see Appendix B). We now look at the operator-state correspondence more closely, starting with some properties of the identity element $\mathbb{1} \in \mathfrak{M}$ and the corresponding endomorphism $\mathbb{1}(z) = \mathrm{id}_V$. First, we note that

$$\mathbb{1} = \lim_{z\to 0} \mathbb{1}(z)\Omega = \mathrm{id}_{\mathfrak{M}}\Omega = \Omega.$$

Also, by definition, $\partial\mathbb{1} = 0$ since

$$(\partial\mathbb{1})(z) = \frac{d}{dz}\mathbb{1}(z) = \frac{d}{dz}\mathrm{id}_{\mathfrak{M}} = 0.$$

Lemma 3.3: $[\mathbb{1}, A]_{n\neq 0} = 0$ *and* $[\mathbb{1}, A]_0 = A$.

Corollary 3.3.1: *It follows from proposition 3.2 and lemma 3.3 that*

$$[T, \mathbb{1}]_2 = [\mathbb{1}, T]_2 + \sum_{l\geq 0} \frac{(-1)^{l+1}}{l!} \partial^l [\mathbb{1}, T]_{l+2} = 0, \tag{3.9}$$

so $\mathbb{1}$ has conformal weight 0.

Let us now look at Jacobi-like identities that the brackets satisfy.

Proposition 3.4: *The brackets satisfy the following identities for all $p, q \in \mathbb{Z}$*

$$[A,[B,C]_p]_q = (-1)^{|A||B|}[B,[A,C]_q]_p + \sum_{l\geq 1}\binom{q-1}{l-1}[[A,B]_l,C]_{p+q-l} \tag{3.10}$$

$$[A,[B,C]_p]_q = (-1)^{|A||B|}\left([B,[A,C]_q]_p - \sum_{l\geq 1}\binom{p-1}{l-1}[[B,A]_l,C]_{p+q-l}\right) \tag{3.11}$$

$$\begin{aligned}[[A,B]_p,C]_q &= \sum_{l\geq q}(-1)^{l-q}\binom{p-1}{l-q}[A,[B,C]_l]_{p+q-1} \\ &\quad - (-1)^{|A||B|+n}\sum_{l\geq 1}(-1)^{p-l}\binom{p-1}{l-1}[B,[A,C]_l]_{p+q-l}.\end{aligned} \tag{3.12}$$

Here, the generalised combinatorial factor is defined $\forall n \in \mathbb{Z}$, $k \in \mathbb{N}$ as

$$\binom{n}{k} := \begin{cases} 0 & \text{if } n = 0,\ k \neq 0 \\ 1 & \text{if } n = 0,\ k = 0 \\ \frac{n(n-1)\dots(n-k+1)}{k!} & \text{otherwise} \end{cases} \tag{3.13}$$

Corollary 3.4.1: *Let $\hat{A} = [A,-]_1 \in \operatorname{End}\mathfrak{M}$. (3.10) implies that*

$$\hat{A}[B,C]_n = [\hat{A}B,C]_n + (-1)^{|A||B|}[B,\hat{A}C]_n. \tag{3.14}$$

Corollary 3.4.1 shows that $[A,-]_1$ is a derivation over the brackets $[-,-]_n$. A special case of this is when A is the Virasoro element and $[T,-]_1 = \partial$. we also have the following:

Corollary 3.4.2: *Let $A = T$, $q = 2$, $[T,B]_2 = h_B B$ and $[T,c]_2 = h_C C$ in (3.10). Then, along with proposition 3.1(a),*

$$[T,[A,B]_n]_2 = (h_A + h_B - n)[A,B]_n. \tag{3.15}$$

Next, we examine the relationship between the brackets of two states and their corresponding modes.

Proposition 3.5: *The vector $[A,B]_n := \lim_{z\to 0}[A,B]_n(z)\mathbb{1}$ can be written as*

$$[A,B]_n = A_{n-h_A}B. \tag{3.16}$$

Definition 12: The **normal-ordered product** $(\ )$: $\mathfrak{M}\otimes\mathfrak{M} \to \mathfrak{M}$ is the $n = 0$ bracket between two states:

$$(AB) := [A,B]_0. \tag{3.17}$$

Proposition 3.6: *The normal-ordered product admits the following properties:*

(a) $(A\mathbb{1}) = (\mathbb{1}A)$

(b) *The modes* $(AB)_n$ *in the expansion of the field* $(AB)(z) = \sum_n (AB)_n z^{-n-h_A-h_B}$ *are given by*

$$(AB)_n = \sum_{l\leq -h_A} A_l B_{n-l} + (-1)^{|A||B|} \sum_{l>-h_A} B_{n-l} A_l. \tag{3.18}$$

Finally, we look at the Lie superalgebra structure on the modes.

Proposition 3.7: *The modes in the expansions* $A(z) = \sum_n A_n z^{-n-h_A}$ *and* $B(z) = \sum_n B_n z^{-n-h_B}$ *satisfy*

$$[A_m, B_n] := A_m B_n - (-1)^{|A||B|} B_n A_m = \sum_{l\geq 1} \binom{n+h_A-1}{l-1} ([A,B]_l)_{m+n} \tag{3.19}$$

This illustrates that the supercommutator of modes depends only on the singular part of the OPE between the corresponding fields. It is worth observing that if we let the states A and B in proposition 3.7 be the Virasoro element, we obtain the Virasoro algebra, keeping in mind the minor caveat that we are regarding $L_n \in \operatorname{End}\mathfrak{M}$, so the Lie bracket is indeed the commutator of linear maps.

The study of string theory via the BRST quantisation procedure requires the addition of Faddeev-Popov ghosts given by primary fields $b(z)$ and $c(z)$ with conformal weights 2 and -1 respectively. This system of ghosts is a special case of what are known collectively as *bc-systems*. We will now study the CFT of bc-systems in general with this VOA.

3.3 BC-systems

Definition 13: A **BC-system** is a CFT containing states $b, c \in V$ with the corresponding fields

$$b(z) = \sum_n b_n z^{-n-\lambda} \qquad c(z) = \sum_n c_n z^{-n-(1-\lambda)}, \tag{3.20}$$

where the modes have the Hermiticity properties[§]

$$b_n^\dagger = \epsilon b_{-n} \qquad c_n^\dagger = c_{-n}. \tag{3.21}$$

Here, $2\lambda \in \mathbb{Z}$ and when λ is a half-integer the modes are labelled either by integers (R-sector) or half-integers (NS-sector). This is summarised as follows:

Sector	Mode	$n \in$
NS	b_n	$\mathbb{Z} - \lambda$
	c_n	$\mathbb{Z} + \lambda$
R	b_n	$\frac{1}{2} + \mathbb{Z} - \lambda$
	c_n	$\frac{1}{2} + \mathbb{Z} + \lambda$

[§]One might notice that this is the opposite sign convention to the one introduced in semi-infinite cohomology in general. Nonetheless, it ensures that the resulting differential that one would obtain is Hermitian.

The non-zero $[-,-]_{n>0}$ brackets are

$$[c,b]_1 = \mathbb{1} \qquad [b,c]_1 = \epsilon\mathbb{1} = \begin{cases} +1 & \text{fermionic } bc\text{-system} \\ -1 & \text{bosonic } bc\text{-system} \end{cases} \tag{3.22}$$

The bosonic bc-system is often referred to as a $\beta\gamma$-system. The Virasoro element/stress tensor takes the form

$$T^{bc} := -\lambda(b\partial c) + (1-\lambda)(\partial bc). \tag{3.23}$$

Using (3.12), we may quickly see that b and c are conformal primaries of weights λ and $1-\lambda$. Using proposition 3.7 gives the (anti-)commutation relations of the modes

$$c_m b_n + \epsilon b_n c_m = \delta_{m+n,0}. \tag{3.24}$$

Proposition 3.8: *T^{bc} indeed admits the brackets of a Virasoro element*

$$[T^{bc},T^{bc}]_4 = \frac{c_{bc}}{2}\mathbb{1} \quad [T^{bc},T^{bc}]_2 = 2T^{bc} \quad [T^{bc},T^{bc}]_1 = \partial T^{bc}, \tag{3.25}$$

where the central charge $c_{bc} = -2\epsilon(6\lambda^2 - 6\lambda + 1) = \epsilon(1-3Q^2)$ and $Q := \epsilon(1-2\lambda)$.

Definition 14: A **vacuum state** $|q\rangle \in V$ of charge q, where $q \in \mathbb{Z} + 1/2$ for the NS sector and $q \in \mathbb{Z}$ for the R sector, is given by the conditions

$$\begin{aligned} b_n\,|q\rangle &= 0 \qquad n > \epsilon q - \lambda \\ c_n\,|q\rangle &= 0 \qquad n \geq -\epsilon q + \lambda \end{aligned} \tag{3.26}$$

There is more that one could say about bc-systems, although we will not go into the details in this report. In particular, one can perform a process called *bosonisation*. By doing so, it can be shown that the various vacua of bosonic bc-systems do not belong to the same representation of the bc mode algebra [22]. For any two vacua $|q\rangle$ and $|q'\rangle$ of a ferminonic bc-system,

$$|q'\rangle = b_{n_1}n_{n_2}\dots b_{n_B}c_{m_1}c_{m_2}\dots c_{m_C}\,|q\rangle\,, \tag{3.27}$$

for some combination of modes (up to rescaling). However, this is never the case in a bosonic bc-system unless $q = q'$. This crucial difference between bosonic and fermionic bc-system underlies the fact that there exists different *pictures* [22] in superstring theories. This is a feature that we will see even with the bosonic Gomis-Ooguri string in the next chapter.

3.4 BRST Quantisation Meets VOAs

Consider some meromorphic CFT described by a VOA over $\mathfrak{M}$ with Virasoro element $T^{\mathfrak{M}}$ such that $2[T^{\mathfrak{M}},T^{\mathfrak{M}}]_4 = c_m\mathbb{1}$ and a fermionic bc-system with $\lambda = 2$. Hence,

$$T^{bc} = -2(b\partial c) - (\partial bc). \tag{3.28}$$

Substituting $\lambda = 2,\ \epsilon = 1$ into 3.8 gives $c_{bc} = -26$. This fermionic bc-system arising from the Faddeev-Popov ghosts in the BRST quantisation procedure is precisely what we encountered when studying the semi-infinite cohomology of the Virasoro algebra (relative to its centre). We can cross-refer equation (3.28) with (2.34) and confirm that the Virasoro element of this bc-system is precisely the generating function $T^{bc}(z)$ introduced in the context of semi-infinite cohomology. We therefore observe that relative semi-infinite cohomology of the Virasoro algebra occurs as a VOA over the space of semi-infinite forms $\Lambda^{\bullet}_{\infty}$. The combined CFT can be described by a VOA over $V^{\bullet} = \mathfrak{M} \otimes \Lambda^{\bullet}_{\infty}$ with Virasoro element $T^{\text{tot}} := T^{\mathfrak{M}} + T^{bc}$ which has central charge $c_{\text{tot}} = c_m + c_{bc}$ since

$$[T^{\text{tot}}, T^{\text{tot}}]_4 = [T^{\mathfrak{M}}, T^{\mathfrak{M}}]_4 + [T^{\mathfrak{M}}, T^{bc}]_4 + [T^{bc}, T^{\mathfrak{M}}]_4 + [T^{bc}, T^{bc}]_4 = c_m + c_{bc}. \tag{3.29}$$

Definition 15: The **BRST current** is defined as

$$j = (cT^{\mathfrak{M}}) + \frac{1}{2}(cT^{bc}). \tag{3.30}$$

The **BRST differential** is constructed from the BRST current

$$d = \oint_{C_0} \frac{dz}{2\pi i} j(z). \tag{3.31}$$

Corollary 3.4.2 tells us that j has conformal weight 1. Hence, its field admits a mode expansion $j(z) = \sum_n j_n z^{-n-1}$, which means we may express the BRST operator as

$$d = j_0 = [j, -]_1 \in \operatorname{End} V. \tag{3.32}$$

From the definition of the BRST current using VOA notation, we see that it is the same $j(z)$ generating function introduced in semi-infinite cohomology. Hence, the BRST differential is the differential from semi-infinite cohomology of the Virasoro algebra. In fact, this can be checked even more explicitly as follows.

Proposition 3.9: *Using the techniques from VOAs, the following expression for d can be obtained:*

$$d = \sum_n c_{-n} L_n^{\mathfrak{M}} + \sum_{\substack{n,m\in\mathbb{Z} \\ n<m}} (n-m) : b_{m+n} c_{-m} c_{-n} : , \tag{3.33}$$

where

$$: b_{m+n} c_{-m} c_{-n} := \begin{cases} b_{m+n} c_{-m} c_{-n}, & \text{for } n+m \le 2 \\ c_{-m} c_{-n} b_{m+n}, & \text{for } n+m > -2 \end{cases} \tag{3.34}$$

This expression in terms of modes is exactly what one would obtain by using $b_n := \iota(L_n)$ and $c_n = \varepsilon(L'_{-n})$ in the definition of the semi-infinite cohomology differential (2.18).

As mentioned in section 2.6, we need $c_m + c_{bc} = 0$ for non-trivial cohomology, so $\mathfrak{M}$ is such that $c_m = 26$.[§]. The classical example is the Fock module of 26 free bosons in string theory but as it stands, $\mathfrak{M}$ can be any Virasoro representation with central charge 26.

[§]In fact, the condition $\lambda = 2$, $D = 26$ can be obtained as a requirement for the BRST operator to square to zero by starting with the BRST current defined for a general fermionic bc-system and then demanding that $[j, j]_1$ is a total derivative.

Let $V^m := \mathfrak{M} \bigotimes \lambda_\infty^m$. Recall from definition 9 that d raises the ghost number n by 1

$$\dots \xrightarrow{d} V^{m-1} \xrightarrow{d} V^m \xrightarrow{d} V^{m+1} \xrightarrow{d} \dots \tag{3.35}$$

Physical states $|\psi\rangle$ in the full space $V = \bigoplus_{m\in\mathbb{Z}} V^m$ are required to be *BRST cocycles*, meaning

$$d\,|\psi\rangle = 0. \tag{3.36}$$

Since $d^2 = 0$, any state of the form

$$|\psi'\rangle = Q\,|\chi\rangle \tag{3.37}$$

for some $|\chi\rangle \in V$ automatically satisfies (3.36). Such states are known as *BRST coboundaries*. However, the BRST coboundaries are orthogonal to all physical states and themselves since

$$\langle\psi|\psi'\rangle = \langle\psi|\,d\,|\chi\rangle = 0 \quad \text{and} \quad \langle\chi|\chi\rangle = \langle\chi|\,d^2\,|\chi\rangle = 0,$$

where we have used $d^\dagger = d$ from proposition 2.8 to obtain the second equation. This implies that any two BRST cocylces $|\psi\rangle$ and $|\psi\rangle + Q\,|\chi\rangle$ that differ by a BRST coboundary are physically indistinguishable since the second term vanishes in all inner products with other BRST cocycles. Thus, the physical states form the **BRST cohomology**

$$H^m_{\text{BRST}} := \frac{\ker\left(d\colon V^m \longrightarrow V^{n+1}\right)}{\operatorname{im}\left(d\colon V^{n-1} \longrightarrow V^m\right)} \qquad H^\bullet_{\text{BRST}} = \sum_{n\in\mathbb{Z}} H^m_{\text{BRST}}. \tag{3.38}$$

Indeed, this is the semi-infinite cohomology of the Virasoro algebra relative to the centre,

$$H^m_{\text{BRST}} = H^m(\text{Vir}, \mathfrak{z}; \mathfrak{M}). \tag{3.39}$$

Recall that there was another grading on the space of semi-infinite forms, deg, that made $\Lambda_\infty^\bullet$ a graded $\mathfrak{g}$-module ($\mathfrak{g} = \text{Vir}$ now). The eigenvalue of L_0^{tot} acts as deg here[§] since we have given ourselves the luxury of working with Vir-modules over which L_0^{tot} is diagonalisable. This leads to another desirable feature in BRST cohomology.

Proposition 3.10:

$$db = T^{\text{tot}} = T^{\mathfrak{M}} + T^{bc} \implies [d, b_0] = db_0 + b_0 d = L_0^{\text{tot}}. \tag{3.40}$$

Proposition 3.10 tells us that L_0^{tot} is BRST exact,

$$[d, L_0^{\text{tot}}] = d(db_0 + b_0 d) - (db_0 + b_0 d)d = 0, \tag{3.41}$$

so d preserves L_0^{tot} eigenspaces. This is consistent with the fact that the differential from semi-infinite cohomology does not change deg. Consider a BRST cocycle $|\phi_n\rangle$ which is an eigenstate of L_0^{tot} with eigenvalue $n \neq 0$. Then

$$L_0^{\text{tot}}\,|\phi_n\rangle = (db_0 + b_0 d)\,|\phi_n\rangle = db_0\,|\phi_n\rangle = n\,|\phi_n\rangle \implies |\phi_n\rangle = d\left(\frac{1}{n} b_0\,|\phi_n\rangle\right). \tag{3.42}$$

[§] Actually, the eigenvalue of L_0^{tot} acts as $-\deg$. This can be shown by a quick calculation of L_0^{tot} acting on monomials spanning $\Lambda_\infty^\bullet$. Likewise deg on $\mathfrak{M}$ can be defined to be compatible with the grading via eigenvalues of L_0^{tot}.

Hence, any BRST cocycle which has a non-zero L_0^{tot} eigenvalue is a BRST coboundary. Since L_0^{tot} is a diagonalisable, any state $|\psi\rangle \in V$ can be decomposed into a linear combination $|\psi\rangle = \sum_n a_n |\phi_n\rangle$, where $a_n \in \mathbb{C}$ (of which only finitely many are non-zero) and $|\phi_n\rangle$ are eigenstates of L_0^{tot} with eigenvalue n. Thus, we may extend the above argument to any cocycle as follows: Any BRST cocycle $|\psi\rangle \in V$ is a linear combination of a $|\phi_0\rangle$ and some BRST coboundaries $|\phi_{n\neq 0}\rangle$. This means that $|\psi\rangle$ and $|\phi_0\rangle$ differ by a BRST coboundary, meaning that they belong to the same equivalence class in cohomology. We therefore see that the cohomology of d acting on V is isomorphic to that of d acting on the subspace spanned by states wtih zero L_0^{tot} eigenvalue. Making contact with our notation from definition 5, we denote L_0^{tot} eigenspaces of eigenvalue n as

$$\left(\mathfrak{M} \otimes \Lambda_\infty^\bullet\right)^n =: V^{\bullet;n} = \oplus_{m\in\mathbb{Z}} V^{m;n}.$$

Then what we have shown is that

$$H^\bullet_{\text{BRST}} \cong H^{\bullet;0}_{\text{BRST}} = \bigoplus_{m\in\mathbb{Z}} H^{m;0}_{\text{BRST}}, \tag{3.43}$$

where

$$H^{m;0}_{\text{BRST}} = \frac{\ker\left(d\colon V^{m;0} \longrightarrow V^{m+1;0}\right)}{\operatorname{im}\left(d\colon V^{m-1;0} \longrightarrow V^{m;0}\right)}. \tag{3.44}$$

It is probably worth mentioning that (3.43) is not a statement of the vanishing theorem (theorem 2.10). First, note that this makes a statement about vanishing cohomology in all non-zero deg, while the vanishing theorem is a statement about vanishing cohomology in all non-zero ghost number. Second, the vanishing theorem was shown for semi-infinite cohomology relative to $\mathfrak{g}_0$, while the BRST cohomology is a special case of semi-infinite cohomology relative to the centre of the Lie algebra.

3.5 A Brief Introduction to $N = 1$ SCFT

It is possible to develop a VOA for the $N = 1$ super-Virasoro algebra. This is the unique associative extension of the Virasoro algebra by a holomorphic primary field $G(z)$ of conformal weight $3/2$ [23]. This is done in detail by Figueroa-O'Farril and Stanciu in the appendix of [24]. We will therefore summarise some of the main ingredients we would need.

We may consider a two-dimensional superconformal field theory in a manifestly supersymmetric form by introducing an anti-commuting variable θ to and defining the action/field content of the theory over a (1|1)-superspace with points $Z := (z, \theta)$. We now have a VOA with states forming a super vector space that is a graded super-Virasoro representation. The corresponding fields are *superfields* $\Phi(Z)$ in the (1|1)-superspace. We may also construct a supercovariant derivative

$$D := \theta\partial + \frac{\partial}{\partial\theta} \implies D^2 = \partial. \tag{3.45}$$

In superspace, there are both even and odd super-intervals, defined as

$$Z_{12} := Z_1 - Z_2 \qquad \theta_{12} := \theta_2 - \theta_1 = Z_{12}^{\frac{1}{2}} \tag{3.46}$$

respectively. Since θ is an anti-commuting variable, the supersymmetric version of Cauchy's residue theorem requires both the contour integral about some z and the Berezin integral over θ. Using this, we may write down OPEs and brackets for superfields $\mathbb{A}(Z)$, $\mathbb{B}(Z)$ as

$$\mathbb{A}(Z_1)\mathbb{B}(Z_2) = \sum_{2r\in\mathbb{Z}} Z_{12}^{-r}[\![A, B]\!]_r(Z_2). \tag{3.47}$$

While we will not get into the details of how to go back and forth between the brackets and "super-contour integrals". However, one useful property of this formulation of a VOA over superspace intervals is that for superfields $\mathbb{A}(Z)$, $\mathbb{B}(Z)$ that split into

$$\mathbb{A}(Z) = \phi_A(z) + \theta\psi_A(z), \quad \mathbb{B}(Z) = \phi_B(z) + \theta\psi_B(z), \tag{3.48}$$

the super-brackets also split into ordinary brackets of the component fields

$$[\![\mathbb{A}, \mathbb{B}]\!]_n = [\phi_A, \phi_B]_n + \theta\big([\psi_A, \phi_B]_n + (-1)^{|\mathbb{A}|}[\phi_A, \psi_B]_n\big) \tag{3.49}$$

$$[\![\mathbb{A}, \mathbb{B}]\!]_{n+\frac{1}{2}} = [\psi_A, \phi_B]_{n+1} - \theta\big([\phi_A, \phi_B]_n + (-1)^{|\mathbb{A}|}[\psi_A, \psi_B]_{n+1}\big) \tag{3.50}$$

for all $n \in \mathbb{Z}$.

As mentioned before, the $N = 1$ super-Virasoro algebra is generated not only by the Virasoro element $T(z)$, but also by a new primary field $G(z)$. The algebra is encoded in the following OPEs

$$\begin{aligned}
T(z)T(w) &= \frac{\frac{c}{2}\mathbb{1}(w)}{(z-w)^4} + \frac{2T(w)}{(z-w)^2} + \frac{\partial T(w)}{z-w} + \text{reg.} \\
T(z)G(w) &= \frac{\frac{3}{2}G(w)}{(z-w)^2} + \frac{\partial G(w)}{z-w} + \text{reg.} \\
G(z)G(w) &= \frac{\frac{2c}{3}\mathbb{1}(w)}{(z-w)^3} + \frac{2T(w)}{(z-w)} + \text{reg.}
\end{aligned} \tag{3.51}$$

Since $G(z)$ has conformal weight $3/2$, we obtain two separate sectors [23] depending on how $G(z)$ behaves under $z \to e^{2\pi i}z$. If $G(e^{2\pi i}z) = G(z)$ then we are in the NS sector, and if $G(e^{2\pi i}z) = -G(z)$, we are in the Ramond sector. The mode expansion $G(z) = G_r z^{-r-3/2}$ is a sum over half-integers in the NS sector and over integers in the R sector. Using proposition 3.7, we may write down the $N = 1$ super-Virasoro algebra in terms of modes as

$$\begin{aligned}
[L_m, L_n] &= L_m L_n - L_n L_m = (m-n)L_{m+n} + \frac{c}{12}m(m^2-1)\delta_{m+n} \\
[L_m, G_r] &= L_m G_r - G_r L_m = (\frac{1}{2}m - r)G_{m+r} \\
[G_r, G_s] &= G_r G_s + G_s G_r = 2L_{r+s} + \frac{c}{3}(r^2 - \frac{1}{4})\delta_{r+s,0}.
\end{aligned} \tag{3.52}$$

A two-dimensional field theory that admits superconformal invariance can be described by a VOA with elements T and G generating the corresponding symmetry algebra. Making contact with our manifestly supersymmetric formulation of the $N = 1$ SCFT VOA, we may assemble $T(z)$ and $G(z)$ into a superfield

$$\mathbb{T}(Z) = \frac{1}{2}G(z) + \theta T(z). \tag{3.53}$$

The construction of this object when we consider the Gomis-Ooguri limit of the NSR string using the relevant matter fields will ensure that worldsheet supersymmetry has been preserved under the limit.

Chapter 4

The Original Gomis-Ooguri String

This chapter introduces the main features of the simplest case of non-relativistic string theory; the Gomis-Ooguri (GO) string [13]. Some results of the paper are derived via a rigorous algebraic approach to BRST quantisation, through which some limitations of the original paper are also are also addressed.

Typically in the worldsheet theory of bosonic strings in D-dimensional Minkowski spacetime, one would start with a collection of scalars

$$X^\mu : \Sigma \longrightarrow \mathbb{R}^{1,D-1},$$

where Σ is the (Euclidean) worldsheet and $\mu \in \{0, 1, \ldots, D-1\}$. These satisfiy the equations of motion

$$\partial\bar{\partial}X^\mu = 0 \implies X^\mu(z,\bar{z}) = X^\mu(z) + \bar{X}^\mu(\bar{z}).$$

Focusing on the holomorphic sector, we have a CFT with Virasoro element (up to some overlooked constant)

$$T = \partial X^\mu \partial X_\mu$$

with central charge $c = D$. This is known as the matter sector of the CFT. The BRST quantisation procedure requires coupling these bosonic matter fields to fermionic bc-ghosts. The ghosts themselves form a fermionic bc-system with $\lambda = 2$ and thus (from (3.23)) admit a VOA with Virasoro element

$$T^{bc} = -2(b\partial c) - (\partial bc).$$

From proposition 3.8, T^{bc} has central charge $c_{bc} = -26$. Thus, the full CFT has a Virasoro element $T_{\text{tot}} = T + T_{bc}$, whose central charge $-26 + D$ needs to vanish for the BRST operator Q to satisfy $Q^2 = 0$. This imposes the well-known condition $D = 26$ in bosonic string theory.

However, the BRST quantisation procedure shows that one only needs the matter sector of the CFT to be some Virasoro module $\mathfrak{M}$ with central charge $c = 26$. The Fock module of 26 free bosons is an example of one such $\mathfrak{M}$, but not the only one. The spectrum of physical states $\mathcal{H}_{\text{phys}}$ is isomorphic to $H^\bullet_\infty(\text{Vir}, \mathfrak{z}; \mathfrak{M})$; the semi-infinite cohomology of the Virasoro algebra relative to the centre, with values in $\mathfrak{M}$. We will demonstrate that the

GO construction of non-relativistic bosonic strings gives rise to a CFT whose matter sector consists of 24 (transverse) free bosons and a $\beta\gamma$-system.

4.1 Setup

Consider the low-energy limit of bosonic string theory

$$S_0 = \frac{1}{4\pi\alpha'}\int d^2\sigma\left(g_{MN}\partial_a X^M \partial_b X^N \eta^{ab} - 2\pi\alpha' B_{MN}\epsilon^{ab}\partial_a X^M \partial_b X^N\right), \tag{4.1}$$

where $M, N \in \{0, 1, \ldots, 25\}$ and the Kalb-Ramond field is such that $B_{01} = B$ and all other components are zero. We define the GO-configuration as the following field configuration

$$2\pi\alpha' B = 1 - \frac{\alpha'}{2\alpha_{eff}}, \qquad g_{\mu\nu} = \eta_{\mu\nu}, \qquad g_{ij} = \frac{\alpha'}{\alpha'_{eff}}\delta_{ij} \tag{4.2}$$

under $\alpha' \to 0$, where $\mu, \nu \in \{0, 1\}$ and $i, j \in \{2, 3, \ldots, 25\}$. Define the coordinates

$$\gamma := X^0 + X^1 \qquad \bar{\gamma} := -X^0 + X^1 \tag{4.3}$$

on the target space and

$$z := e^{i(\sigma^0+\sigma^1)} \qquad \bar{z} := e^{i(\sigma^0-\sigma^1)} \tag{4.4}$$

on the worldsheet. Letting $\partial := \partial/\partial z$ and $\bar{\partial} := \partial/\partial\bar{z}$, we have

$$\frac{\partial}{\partial\sigma_0} = \frac{\partial z}{\partial\sigma_0}\partial + \frac{\partial\bar{z}}{\partial\sigma_0}\bar{\partial} = i(z\partial + \bar{z}\bar{\partial}) \tag{4.5}$$

$$\frac{\partial}{\partial\sigma_1} = \frac{\partial z}{\partial\sigma_1}\partial + \frac{\partial\bar{z}}{\partial\sigma_0}\bar{\partial} = i(z\partial - \bar{z}\bar{\partial}). \tag{4.6}$$

Also,

$$d^2z := dz \wedge d\bar{z} = 2z\bar{z}d\sigma^0 \wedge d\sigma^1 = 2z\bar{z}d^2\sigma \iff d^2\sigma = \frac{1}{2z\bar{z}}d^2z. \tag{4.7}$$

Substituting (4.5), (4.6) and (4.7) into (4.1) gives

$$S_0 = -\frac{1}{4\pi\alpha'}\int d^2z\big(\partial\gamma\bar{\partial}\bar{\gamma} + \partial\bar{\gamma}\bar{\partial}\gamma - 2\pi\alpha' B(\partial\gamma\bar{\partial}\bar{\gamma} - \partial\bar{\gamma}\bar{\partial}\gamma) + 2g_{ij}\partial X^i\bar{\partial}X^j\big). \tag{4.8}$$

Performing Wick rotations in both the worldsheet and target space

$$\sigma^0 \to i\sigma^0 \qquad X^0 \to iX^0$$

we obtain the Euclidean action

$$S_0 = \frac{1}{4\pi\alpha'}\int d^2z\big(\partial\gamma\bar{\partial}\bar{\gamma} + \partial\bar{\gamma}\bar{\partial}\gamma - 2\pi\alpha' B(\partial\gamma\bar{\partial}\bar{\gamma} - \partial\bar{\gamma}\bar{\partial}\gamma) + 2g_{ij}\partial X^i\bar{\partial}X^j\big). \tag{4.9}$$

Introducing Lagrange multipliers β and $\bar{\beta}$ gives the classically equivalent action

$$S_1 = \int \frac{d^2z}{2\pi}\big(\beta\bar{\partial}\gamma + \bar{\beta}\partial\bar{\gamma} - \frac{2\alpha'}{1+2\pi\alpha' B}\bar{\beta}\beta + \frac{1-2\pi\alpha' B}{2\alpha'}\partial\gamma\bar{\partial}\bar{\gamma} + \frac{1}{\alpha'}g_{ij}\partial X^i\bar{\partial}X^j\big). \tag{4.10}$$

Substituting the equations of motion

$$\frac{\delta S_1}{\delta \beta} = 0 \iff \bar{\beta} = \frac{1 + 2\pi\alpha' B}{2\alpha'}\bar{\partial}\gamma \tag{4.11}$$

$$\frac{\delta S_1}{\delta \bar{\beta}} = 0 \iff \beta = \frac{2\alpha'}{1 + 2\pi\alpha' B}\partial\bar{\gamma} \tag{4.12}$$

into S_1 allows us to recover S_0, thereby implying classical equivalence of S_0 and S_1. Taking the GO-limit in (4.10) gives

$$S_{GO} = \int \frac{d^2z}{2\pi}\big(\underbrace{\beta\bar{\partial}\gamma + \bar{\beta}\partial\bar{\gamma}}_{\beta\gamma\text{-system}} + \underbrace{\frac{1}{\alpha'}\delta_{ij}\partial X^i\bar{\partial}X^j}_{\text{24 transverse free bosons}} + \underbrace{\frac{1}{4\alpha'_{eff}}\partial\gamma\bar{\partial}\bar{\gamma}}_{\text{worldsheet instanton}}\big). \tag{4.13}$$

The action S_{GO} describes the CFT of a bosonic $\beta\gamma$-system and 24 free bosons, with $\gamma : \Sigma \to \mathbb{R}^2$ a worldsheet instanton (insert footnote: contact terms in OPEs). The equations of motion force β, γ to be holomorphic and $\bar{\beta}, \bar{\gamma}$ to be anti-holomorphic, so we focus on the holomorphic sector for simplicity. We infer that in the GO-limit, two of the 26 free bosons get replaced by a $\beta\gamma$-system with $\lambda = 1$. Inserting $\epsilon = -1$, $\lambda = 1$ into proposition 3.8 tells us that the $\beta\gamma$-system still contributes the same central charge (of 2) to the matter sector of the CFT as two free bosons, which is sensible, since we have not explicitly changed the matter content of our theory, and is in keeping with the requirement that the matter CFT has a central charge of 26. We now proceed with the computation of the closed string spectrum.

4.2 Compactification and Virasoro Modes

In the language of VOAs, our matter CFT is described by a VOA with Virasoro element $T^{\mathfrak{M}} = T^{\beta\gamma} + T^X$, where T^X is the Virasoro element of 24 free bosons $\{X^i\}_{i\in\{2,\ldots 25\}}$. First let X^1 be non-compact. Then $\beta(z)$ and $\gamma(z)$ admit the usual mode expansions

$$\beta(z) = \sum_{n\in\mathbb{Z}} \beta_n z^{-n-1} \qquad \gamma(z) = \sum_{n\in\mathbb{Z}} \gamma_n z^{-n}.$$

Note that there are no separate NS and R sectors here since λ is an integer. Gomis and Ooguri have shown that one does not obtain any non-trivial spectrum apart from the tachyon when X^1 is non-compact [13]. Hence, we will only consider the case where X^1 is compactified on a circle of radius R. $\gamma(z)$ acquires a non-trivial winding sector [13], reflected by the mode expansion

$$\gamma(z) = iwR\log z + \sum_{n\in\mathbb{Z}} \gamma_n z^{-n}, \tag{4.14}$$

where $w \in \mathbb{Z}$ is the winding number. Applying proposition 3.1 on (4.14) implies that

$$\partial\gamma(z) := -iwRz^{-1} + \sum_{n\in\mathbb{Z}} -n\gamma_n z^{-n-1} \iff (\partial\gamma)_{n\neq 0} = -n\gamma_n \quad (\partial\gamma)_0 = iwR. \tag{4.15}$$

We drop the normal ordered brackets from this point on, so $AB = (AB)$ for two states A and B. A normal ordered product of multiple states is given by

$$A_1A_2A_3 = (A_1(A_2A_3)).$$

From (3.23), we have $T^{\beta\gamma} = -\beta\partial\gamma$, from which we obtain an expression for $L_n^{\beta\gamma}$ in terms of β_n and γ_n

$$\begin{aligned} L_n^{\beta\gamma} &= -\sum_{l\leq -1} \beta_l(\partial\gamma)_{n-l} - \sum_{l>-1} (\partial\gamma)_{n-l}\beta_l && \text{(proposition 3.6(b))} \\ &= \sum_{l\leq -1} (n-l)\beta_l\gamma_{n-l} + \sum_{l>-1} (n-l)\gamma_{n-l}\beta_l - iwR\beta_l && \text{(equation (4.15))} \\ &= \sum_{l\in\mathbb{Z}} (n-l) :\beta_l\gamma_{n-l}: -iwR\beta_l. \end{aligned}$$

Relabelling the summation with $k = n - l$ gives

$$L_n^{\beta\gamma} = \sum_{k\in\mathbb{Z}} k :\beta_{n-k}\gamma_k: . \tag{4.16}$$

The normal ordering of modes (denoted by the colons) is with respect to the vacuum $|0\rangle_{\beta\gamma}$ which, from (3.26), is annihilated by $\beta_{n\geq 0}$ and $\gamma_{n\geq 1}$. Likewise, we know from proposition 2.11 in semi-infinite cohomology (but also see Appendix B for a derivation using VOAs) that

$$L_n^{bc} = \sum_{k\in\mathbb{Z}} (k-n) : c_{-n}b_{k+n}. \tag{4.17}$$

In particular, note the expression for L_0^{tot}:

$$L_0^{\text{tot}} = -iwR\beta_0 + \sum_{k\in\mathbb{Z}} k :\beta_{-k}\gamma_k: + \sum_{k\in\mathbb{Z}} k : c_{-n}b_n. \tag{4.18}$$

The first time is only present when there is winding, so it is absent in the case where X^1 is non-compact.

4.3 Constructing the BRST Cohomology

We start by constructing the BRST differential

$$d = j_0 = (cT^{\beta\gamma})_0 + (cT^X)_0 + \frac{1}{2}(cT^{bc})_0. \tag{4.19}$$

Once again, using VOAs, we have

$$
\begin{aligned}
(cT^X)_0 &= \sum_{l\le 1} c_l L^X_{-l} + \sum_{l>1} L^X_{-l} c_l = \sum_{l\in\mathbb{Z}} c_{-l} L^X_l \\
(cT^{\beta\gamma})_0 &= \sum_{l\le 1} c_l L^{\beta\gamma}_{-l} + \sum_{l>1} L^{\beta\gamma}_{-l} c_l \\
&= \sum_{l\le 1}\sum_{k\in\mathbb{Z}} k c_l : \beta_{-l-k}\gamma_k : + \sum_{l>1}\sum_{k\in\mathbb{Z}} k : \beta_{-l-k}\gamma_k : c_l \\
&- iwR \sum_{l\le 1} c_l \beta_{-l} - iwR \sum_{l>1} \beta_{-l} c_l \\
&= -iwR \sum_{l\in\mathbb{Z}} c_{-l}\beta_l - \sum_{l,k\in\mathbb{Z}} k : c_{-l}\beta_{l-k}\gamma_k : \ .
\end{aligned}
\tag{4.20}
$$

A more lengthy calculation (see proof of proposition 3.9 in Appendix B) shows that

$$
\frac{1}{2}(cT^{bc})_0 = \sum_{\substack{l,k\in\mathbb{Z}\\ k<l}} (k-l) : b_{k+l} c_{-l} c_{-k} : \ . \tag{4.21}
$$

Putting these together, we get

$$
d = -iwR \sum_{l\in\mathbb{Z}} c_{-l}\beta_l + \sum_{\substack{l\in\mathbb{Z}\\ k\in\mathbb{Z}}} k : c_{-l}\beta_{l-k}\gamma_k : + \sum_{l\in\mathbb{Z}} c_{-l} L^X_l + \sum_{\substack{l,k\in\mathbb{Z}\\ k<l}} (k-l) : b_{k+l} c_{-l} c_{-k} : \ . \tag{4.22}
$$

Now, we would like to compute the BRST cohomology of the complex

$$
\dots \xrightarrow{d} V^{m-1} \xrightarrow{d} V^m \xrightarrow{d} V^{m+1} \xrightarrow{d} \dots, \tag{4.23}
$$

where $V = \mathfrak{M} \bigotimes \Lambda^{\bullet}_{\infty}$ and $V^m = \mathfrak{M} \bigotimes \Lambda^m_{\infty}$. Here, $\mathfrak{M} = V_{\beta\gamma} \bigotimes \mathcal{F}(p)$, where $V_{\beta\gamma}$ is the space of states spanned by monomials formed by modes β_n and γ_n acting on some choice of vacuum $|q\rangle_{\beta\gamma}$ and $\mathcal{F}(p)$ is the Fock module of 24 free bosons with vacuum $|p\rangle$ of momentum p^i.

4.3.1 Filtering the Cochain Complex

With the desire of using spectral sequences (see Appendix A for a review), we would like to apply a filtration on V. We assign a filtration degree fdeg to each mode that appears in d.

Mode	β_n	γ_n	b_n	c_n	L^X_n
fdeg	-1	1	-1	1	0

This assignation is compatible with the mode algebra of fermionic and bosonic bc-systems. Then a decreasing filtration F on V can be defined as

$$
F^p V := \{v \in V \mid \operatorname{fdeg} v \ge p\}. \tag{4.24}
$$

This splits d into

$$d = -iwRd_0 + d_1, \tag{4.25}$$

where

$$d_0 = \sum_{l\in\mathbb{Z}} c_{-l}\beta_l \tag{4.26}$$

$$d_1 = \sum_{\substack{l\in\mathbb{Z}\\ k\in\mathbb{Z}}} k : c_{-l}\beta_{l-k}\gamma_k : + \sum_{l\in\mathbb{Z}} c_{-l}L_l^X + \sum_{\substack{l,k\in\mathbb{Z}\\ k<l}} (k-l) : b_{k+l}c_{-l}c_{-k} : \tag{4.27}$$

d preserves the filtration, so $d(F^pV) \subseteq F^pV$. d_r is the part of d that raises fdeg by r, for $r \in \{0,1\}$. The condition $d^2 = 0 \iff d_0^2 = d_1^2 = 0$ and $d_0d_1 + d_1d_0 = 0$.

Additionally, this filtration is exhaustive and weakly convergent since

$$\bigcup_{p\in\mathbb{Z}} F^pV = V \quad \text{and} \quad \bigcap_{p\in\mathbb{Z}} F^pV = 0. \tag{4.28}$$

Thus, theorem A.1 tells us that there exists a spectral sequence with

$$E_1^{p,q} \cong H^{p+q}\left({}^{F^pV}\big/_{F^{p+1}V}, d_0 \right) \tag{4.29}$$

and converges to $H^\bullet(V,d)$,

$$E_\infty^{p,q} \cong E_0^{p,q}\big(H^\bullet(V,d), F\big) = {}^{F^pH^{p+q}(V,d)}\big/_{F^{p+1}H^{p+q}(V,d)}\,. \tag{4.30}$$

Using the *Kugo-Ojima (KO) quartet mechanism* [25], we aim to show that $d_1 \equiv 0$ on the cohomology of d_0, $H_{d_0} = E_1^{\bullet,\bullet}$. Thus, the spectral sequence collapses and we have

$$H^\bullet(V,d) \cong E_1^{\bullet,\bullet} = H_{d_0}. \tag{4.31}$$

4.3.2 The Kugo-Ojima Quartet Mechanism

To understand how the KO mechanism works, let us consider a simpler case before looking at the GO-string. Let there be two sets of creation and annihilation operators, $(a^\dagger, a)$ and $(\mathscr{a}^\dagger, \mathscr{a})$, acting on some vector space of states generated by the creation operators[§] $a^\dagger$ and $\mathscr{a}^\dagger$ acting on a vacuum state $|0\rangle$. The two sets of operators obey the following commutation relations

$$\begin{gathered}
[a, a^\dagger] = 1 \iff aa^\dagger = 1 + a^\dagger a \\
[\mathscr{a}, \mathscr{a}^\dagger]_+ = 1 \iff \mathscr{a}\mathscr{a}^\dagger = 1 - \mathscr{a}^\dagger\mathscr{a} \\
[a, a] = [a^\dagger, a^\dagger] = [\mathscr{a}, \mathscr{a}]_+ = [\mathscr{a}^\dagger, \mathscr{a}^\dagger]_+ = 0 \\
[a, \mathscr{a}] = [a, \mathscr{a}^\dagger] = [a^\dagger, \mathscr{a}^\dagger] = 0
\end{gathered}$$

§The dagger here does not denote Hermitian conjugation. It is simply notation to denote a creation operator. This point will be clarified when we look at the KO mechanism for the GO-string.

and the vacuum obeys

$$a |0\rangle = \not{a} |0\rangle = 0. \tag{4.32}$$

Let $Q = a^\dagger \not{a}$. We would like compute the cohomology of Q acting on such a space. However, it turns out that this is isomorphic to the cohomology of Q acting on a smaller subspace. To see how this occurs, we construct the operator

$$K := \not{a}^\dagger a. \tag{4.33}$$

Then

$$\begin{aligned} QK + KQ &= a^\dagger \not{a} \not{a}^\dagger a + \not{a}^\dagger a a^\dagger \not{a} \\ &= (1 - \not{a}^\dagger \not{a}) a^\dagger a + \not{a}^\dagger \not{a}(1 + a^\dagger a) \\ &= a^\dagger a + \not{a}^\dagger \not{a} \\ &=: N_{tot}. \end{aligned} \tag{4.34}$$

N_{tot} is called the *number operator*, since it counts the number of creation operators acting on $|0\rangle$

$$\begin{aligned} N_{tot}(a^\dagger |0\rangle) &= a^\dagger(1 + a^\dagger a) |0\rangle = a^\dagger |0\rangle \\ N_{tot}(\not{a}^\dagger |0\rangle) &= \not{a}^\dagger(1 - \not{a}^\dagger \not{a}) |0\rangle = \not{a}^\dagger |0\rangle . \end{aligned} \tag{4.35}$$

Since we had postulated that the space of states is spanned by such monomials, we may decompose any state $|\psi\rangle = \sum_n |\phi_n\rangle$, where $N_{tot} |\phi_n\rangle = n |\phi_n\rangle$. Thus, we can construct the same argument we did to show that BRST cohomology resides in the zero-conformal weight subspace of the full space, to show that the Q-cohomology of this space resides only in the subspace with zero N_{tot} eigenvalue. This is because any $|\phi_{n\neq 0}\rangle$ which is a Q-cocycle is inevitably a Q-coboundary since

$$N_{tot} |\phi_n\rangle = n |\phi_n\rangle = (QK + KQ) |\phi_n\rangle = QK |\phi_n\rangle \iff |\phi_n\rangle = Q\left(\frac{1}{n} K |\phi_n\rangle\right). \tag{4.36}$$

Consequently, any general Q-cocycle $|\psi\rangle$ may be decomposed into a linear combination $|\psi\rangle = |\phi_0\rangle + |\chi\rangle$, where $Q |\phi_0\rangle = 0$ and $|\chi\rangle = \sum_{n\neq 0} |\phi_n\rangle = \sum_{n\neq 0} Q\left(\frac{1}{n} |\phi_n\rangle\right)$ is a Q-coboundary. But the zero-N_{tot} eigenvalue subspace is spanned only by $|0\rangle$, which means $|\phi_0\rangle = k |0\rangle$, for some $k \in \mathbb{C}$. Hence,

$$H_Q \cong \mathbb{C} |0\rangle . \tag{4.37}$$

Let us now see how we may exploit such a construction for the space of states $\mathcal{V} = V_{\beta\gamma} \otimes \Lambda^{\bullet}_{\infty}$ spanned by the $\beta\gamma$ and bc-systems in the GO-string setting.

4.3.3 The GO-String Spectrum

Recall that from our spectral sequence argument, we start by computing the cohomology of d_0 acting on $E_0^{p,q}$, the bigraded Vir-module associated to the filtration F defined as

$$E_0^{p,q} := F^p V^{p+q} \big/ F^{p+1} V^{p+q} . \tag{4.38}$$

By construction of F in (4.24),

$$E_0^{p,q} = \{v \in V^{p+q} \mid \operatorname{fdeg} v = p\}. \tag{4.39}$$

Thus, we see that $E_0 = \bigoplus_{p,q} E_0^{p,q} = V$, and since the action of d_0 is given by

$$\dots \xrightarrow{d_0} E_0^{p,q-1} \xrightarrow{d_0} E_0^{p,q} \xrightarrow{d_0} E_0^{p,q+1} \xrightarrow{d_0} \dots, \tag{4.40}$$

for each fdeg, we may consider the cohomology of the chain complex

$$\dots \xrightarrow{d_0} V^{m-1} \xrightarrow{d_0} V^{m} \xrightarrow{d_0} V^{m+1} \xrightarrow{d_0} \dots. \tag{4.41}$$

From (4.26), we infer that d_0 only acts non-trivially on $\mathcal{V}$, so the chain complex we are working with is

$$\dots \xrightarrow{d_0} \mathcal{V}^{m-1} \xrightarrow{d_0} \mathcal{V}^{m} \xrightarrow{d_0} \mathcal{V}^{m+1} \xrightarrow{d_0} \dots, \tag{4.42}$$

where $\mathcal{V}^m = V_{\beta\gamma} \bigotimes \Lambda^m_\infty$.

To make use of the KO mechanism, we want to write d_0 in a form similar to that of Q in the previous subsection. In this case, instead of two pairs of creation and annihilation operators, one bosonic and one fermionic, there are four pairs, with each pair labelled by an integer. Two of them are bosonic, corresponding to the modes β_n and γ_n, while the other two are fermionic, corresponding to the modes b_n and c_n. However, before we can write down these sets of operators, we need to know which vacuum $|p\rangle_{bc} \otimes |q\rangle_{\beta\gamma}$ to choose for $\mathcal{V}$. Recall from 3.3 that a general vacuum state of a (bosonic or fermionic) bc-system is annihilated by a set of modes given by (3.26). For the GO-string, the condition that $L_0^{\text{tot}} |phys\rangle = 0$ requires $\beta_0 |q\rangle_{\beta\gamma} \neq 0$. Otherwise, the contribution from the winding sector in (4.18) would make no difference and we would be left with a trivial spectrum barring the tachyon. This imposes the condition $q \leq -1$ on the $\beta\gamma$-vacuum $|q\rangle_{\beta\gamma}$. Next, we need a "vacuum matching" condition for the KO mechanism to work. That is, if $\beta_n |q\rangle_{\beta\gamma} = 0$ for some $n \in \mathbb{Z}$, then $b_n |p\rangle_{bc} = 0$ too, and likewise for γ_n and c_n acting on the respective vacua. From (3.26), this demands

$$p - 2 = -q - 1 \implies p = -q + 1.$$

Hence, the allowed vacua are labelled by $m \in \mathbb{N}^+$:

$$|\text{vac}\rangle_m = |-m\rangle_{\beta\gamma} \otimes |m+1\rangle_{bc}\,. \tag{4.43}$$

We now split d_0 as

$$d_0 = \sum_{n\in\mathbb{Z}} c_{-n}\beta_n = \sum_{n\geq m} c_{-n}\beta_n + \sum_{n\geq 1-m} \beta_{-n} c_n. \tag{4.44}$$

This ensures that annihilators of $|\text{vac}\rangle_m$ are placed to the right. We now come back to the four sets of creation and annihilation operators as promised and we write down an explicit assignation of creation and annihilation operators to each mode $\beta_n, \gamma_n, b_n, c_n$, summarised below:

$$\begin{aligned}
&a^\dagger(\beta_n) = \beta_{-n} && n \geq 1-m && a^\dagger(\gamma_n) = -\gamma_{-n} && n \geq m \\
&a(\beta_n) = \gamma_n && n \geq 1-m && a(\gamma_n) = \beta_n && n \geq m \\
&a^\dagger(b_n) = b_{-n} && n \geq 1-m && a^\dagger(c_n) = c_{-n} && n \geq m \\
&a(b_n) = c_n && n \geq 1-m && a(c_n) = b_n && n \geq m.
\end{aligned} \tag{4.45}$$

These operators are indeed compatible with the mode algebra of the $\beta\gamma$ and bc-systems, which can be expressed as

$$\begin{aligned}
[a(\phi_k), a^\dagger(\phi_l)] &= \delta_{kl} \qquad \phi = \beta, \gamma \\
[a(\phi_k), a^\dagger(\phi_l)]_+ &= \delta_{kl} \qquad \phi = b, c
\end{aligned} \tag{4.46}$$

Then using (4.45),

$$d_0 = \sum_{n \geq m} a^\dagger(c_n) a(\gamma_n) + \sum_{n \geq 1-m} a^\dagger(\beta_n) a(b_n).$$

Just as we did in the previous subsection, we may construct an operator K from the expression above for d_0

$$K = \sum_{n \geq m} a^\dagger(\gamma_n) a(c_n) + \sum_{n \geq 1-m} a^\dagger(b_n) a(\beta_n). \tag{4.47}$$

Finally, we may construct the analogous number operator

$$\begin{aligned}
N_{tot} &= d_0 K + K d_0 \\
&= \sum_n a^\dagger(\beta_n) a(\beta_n) + \sum_n a^\dagger(\gamma_n) a(\gamma_n) + \sum_n a^\dagger(b_n) a(b_n) + \sum_n a^\dagger(c_n) a(c_n).
\end{aligned} \tag{4.48}$$

The space $\mathcal{V}$ is spanned by monomials of the form

$$\omega := \beta_{-m_1}\beta_{-m_2}\ldots\beta_{-m_{\mathcal{B}}}\gamma_{-n_1}\gamma_{-n_2}\ldots\gamma_{-n_\Gamma} b_{-l_1} b_{-l_2}\ldots b_{-l_B} c_{-k_1} c_{-k_2}\ldots c_{-k_C} \left|\text{vac}\right\rangle_m, \tag{4.49}$$

where $\mathcal{B}$, Γ, B and C are the number of β_n, γ_n, b_n and c_n respectively, and

$$\begin{gathered}
m_1 \geq m_2 \geq \cdots \geq m_{\mathcal{B}} \geq 0 \\
n_1 \geq n_2 \geq \cdots \geq n_\Gamma \geq 1 \\
l_1 > l_2 > \cdots > l_B > 0 \\
k_1 > k_2 \cdots > k_C \geq 1.
\end{gathered} \tag{4.50}$$

Writing (4.49) in terms of the creation operators in (4.45), it can be quickly checked that the action of N_{tot} on (4.49) is indeed

$$N_{tot}\omega = \big(\mathcal{B} + \Gamma + B + C\big)\omega, \tag{4.51}$$

for any monomial ω. Thus, the eigenvalue of N_{tot} grades the space $\mathcal{V} = \bigoplus_{n\in\mathbb{N}} \mathscr{V}^n$, where $N_{tot}v = nv \ \forall v \in \mathscr{V}^n$ and $\mathscr{V}^0 \cong \mathbb{C}\left|\text{vac}\right\rangle_m$. Consequently, by the KO mechanism, we now have that

$$H_{d_0} \cong \mathbb{C}\left|\text{vac}\right\rangle_m \cong E_1 \cong E_1^{0,0} \cong H_{d_0}^{0,0}. \tag{4.52}$$

Page 2 of the spectral sequence is the d_1-cohomology on E_1. However, from (4.52), we observe that d_1 must be zero on $E_1^{\bullet,\bullet}$ since $E_1^{p,q} = 0$ for all $p, q \neq 0$, while $d_1 : E_1^{p,q} \to E_1^{p+1,q}$. To see this in more detail, let us consider the action of d_1 on representatives of d_0-cohomology in $E_0^{p,q}$. We started with the following diagram.

$$
\begin{tikzcd}
 & \vdots & \vdots & \vdots & \\
\dots \arrow[r, "d_0"] & F^{p-1}\mathcal{V}^{n-1} \arrow[u, hook] \arrow[r, "d_0"] & F^{p-1}\mathcal{V}^{n} \arrow[u, hook] \arrow[r, "d_0"] & F^{p-1}\mathcal{V}^{n+1} \arrow[u, hook] \arrow[r, "d_0"] & \dots \\
\dots \arrow[r, "d_0"] & F^{p}\mathcal{V}^{n-1} \arrow[u, hook] \arrow[r, "d_0"] & F^{p}\mathcal{V}^{n} \arrow[u, hook] \arrow[r, "d_0"] & F^{p}\mathcal{V}^{n+1} \arrow[u, hook] \arrow[r, "d_0"] & \dots \\
\dots \arrow[r, "d_0"] & F^{p+1}\mathcal{V}^{n-1} \arrow[u, hook] \arrow[r, "d_0"] & F^{p+1}\mathcal{V}^{n} \arrow[u, hook] \arrow[r, "d_0"] & F^{p+1}\mathcal{V}^{n+1} \arrow[u, hook] \arrow[r, "d_0"] & \dots \\
 & \vdots \arrow[u, hook] & \vdots \arrow[u, hook] & \vdots \arrow[u, hook] &
\arrow[from=2-2, to=3-3, "d_1"]
\arrow[from=2-3, to=3-4, "d_1"]
\arrow[from=3-2, to=4-3, "d_1"]
\arrow[from=3-3, to=4-4, "d_1"]
\end{tikzcd}
$$

Figure 4.1: A depiction of the filtered differential structure, the basis for our spectral sequence.

Letting $n = p + q$ for clarity and taking quotients along the inclusions (where we use (4.38)), we obtain the double complex $\{E_0^{\bullet,\bullet}, d_0, d_1\}$.

$$
\begin{tikzcd}
 & \vdots \arrow[d, "d_1"] & \vdots \arrow[d, "d_1"] & \vdots \arrow[d, "d_1"] & \\
\dots \arrow[r, "d_0"] & E_0^{p-1,q-1} \arrow[r, "d_0"] \arrow[d, "d_1"] & E_0^{p-1,q} \arrow[r, "d_0"] \arrow[d, "d_1"] & E_0^{p-1,q+1} \arrow[r, "d_0"] \arrow[d, "d_1"] & \dots \\
\dots \arrow[r, "d_0"] & E_0^{p,q-1} \arrow[r, "d_0"] \arrow[d, "d_1"] & E_0^{p,q} \arrow[r, "d_0"] \arrow[d, "d_1"] & E_0^{p,q+1} \arrow[r, "d_0"] \arrow[d, "d_1"] & \dots \\
\dots \arrow[r, "d_0"] & E_0^{p+1,q-1} \arrow[r, "d_0"] \arrow[d, "d_1"] & E_0^{p+1,q} \arrow[r, "d_0"] \arrow[d, "d_1"] & E_0^{p+1,q+1} \arrow[r, "d_0"] \arrow[d, "d_1"] & \dots \\
 & \vdots & \vdots & \vdots &
\end{tikzcd}
$$

Figure 4.2: The double complex structure on the zeroth page of our spectral sequence.

As outlined earlier, the explicit bigrading is just for brevity. For example, $E_0^{p,q}$ denotes elements of $\mathcal{V}^{p+q}$ with fdeg $= p$. When we omit the bigrading, we refer to $\mathcal{V}$ as a whole, since a sum over fdeg and ghost number of $E_0^{p,q}$ yields $\mathcal{V}$ anyway. In figure 4.2, the bigrading highlights the fact that the d_0 chain complex splits into one chain complex for each fdeg, while d_1 behaves like a chain map between these chain complexes of various fdeg.

Now letting $p = q = 0$, note that d_0-cohomology is non-trivial only at $E_0^{0,0}$:

$$\begin{array}{ccccccccc}
 & & \vdots & & \vdots & & \vdots & & \\
 & & \downarrow{\scriptstyle d_1} & & \downarrow{\scriptstyle d_1} & & \downarrow{\scriptstyle d_1} & & \\
\cdots & \xrightarrow{d_0} & E_0^{-1,q-1} & \xrightarrow{d_0} & E_0^{-1,0} & \xrightarrow{d_0} & E_0^{-1,1} & \xrightarrow{d_0} & \cdots \\
 & & \downarrow{\scriptstyle d_1} & & \downarrow{\scriptstyle d_1} & & \downarrow{\scriptstyle d_1} & & \\
\cdots & \xrightarrow{d_0} & E_0^{0,-1} & \xrightarrow{d_0} & E_0^{0,0} & \xrightarrow{d_0} & E_0^{0,1} & \xrightarrow{d_0} & \cdots \\
 & & \downarrow{\scriptstyle d_1} & & \downarrow{\scriptstyle d_1} & & \downarrow{\scriptstyle d_1} & & \\
\cdots & \xrightarrow{d_0} & E_0^{1,-1} & \xrightarrow{d_0} & E_0^{1,0} & \xrightarrow{d_0} & E_0^{1,1} & \xrightarrow{d_0} & \cdots \\
 & & \downarrow{\scriptstyle d_1} & & \downarrow{\scriptstyle d_1} & & \downarrow{\scriptstyle d_1} & & \\
 & & \vdots & & \vdots & & \vdots & &
\end{array}$$

Figure 4.3: Apart from $E_0^{0,0}$ which contains $|\text{vac}\rangle_m$, every other term in this page of the spectral sequence is spanned by states with non-zero N_{tot}-eigenvalue, which means they are spanned by d_0-coboundaries.

Thus, the d_1 chain complexes reduce to the following (which are now exact sequences)

$$\cdots \xrightarrow{d_1} H_{d_0}^{k-1,0} \cong 0 \xrightarrow{d_1} H_{d_0}^{k,0} \xrightarrow{d_1} H_{d_0}^{k+1,0} \cong 0 \xrightarrow{d_1} \cdots \tag{4.53}$$

which enforce $d_1 \equiv 0$ for all $k \in \mathbb{Z}$. Another way to show this is to look at figure 4.3 and recall that the zero N_{tot} eigenvalue space $\mathscr{V}^0 \cong \mathbb{C}\,|\text{vac}\rangle_m \subset E_0^{0,0}$. Hence, every $E_0^{k,l}$ is spanned by d_0-coboundaries for k or l not equal to zero. It therefore follows that $d_1 v$ is d_0-coboundary $\forall v \in \mathscr{V}$. More precisely,

$$\operatorname{im}\left(d_1 : E_0^{\bullet,\bullet} \to E_0^{\bullet,\bullet}\right) \subset \operatorname{im}\left(d_0 : E_0^{\bullet,\bullet} \to E_0^{\bullet,\bullet}\right).$$

This statement is equivalent to the fact that $d_1 \equiv 0$ on H_{d_0}, since d_0-coboundaries belong to the trivial class in H_{d_0}. Thus,

$$H_{d_1}(H_{d_0}) \cong E_2 = 0 \tag{4.54}$$

so the spectral sequence indeed collapses (see Appendix A),

$$E_\infty \cong H(\mathcal{V}, d) \cong H_{d_0}. \tag{4.55}$$

We may thereby deduce that the Hilbert space of physical states of the bosonic GO-string is

$$\mathcal{H}_{\text{phys}} \cong \mathcal{H}_X \bigotimes H_{d_0} \cong \mathcal{H}_X \bigotimes \mathbb{C}\,|\text{vac}\rangle_m\,. \tag{4.56}$$

Each choice of $m \in \mathbb{N}^+$ gives rise to isomorphic BRST cohomology. Hence, there exists an infinite number of pictures.[§] Additionally, notice that the 24 free bosons seemed to just be carried along for the ride. In more formal language, while we know that $\mathcal{H}_X$ is the Hilbert space of the Fock module of 24 bosons, we never used any explicit knowledge of this (notice that we never needed an expression for its representation, meaning we never actually used L_n^X in terms of its modes) at any point. It is natural to ask what other $c = 24$ Virasoro modules could take its place instead and what this could imply physically, such as what spectrum this would give or whether such a theory is still Galilean invariant etc.

[§]These are related by a picture-changing operator, which Narganes showed to be an isomorphism in a long exact sequence of cohomology, induced by the existence of a chain map [26].

Chapter 5

The NSR Gomis-Ooguri String

The non-relativistic GO-limit of the bosonic string singled out the timelike target space coordinate X^0 and a spacelike coordinate (chosen to be X^1 for specificity). From the worldsheet perspective, these two free bosons became a $\beta\gamma$-system under the GO-limit. Now let us consider the Neveu-Schwarz Ramond (NSR) string. Its matter content is comprised 10 free bosons X^μ and 10 free fermions ψ^μ, coupled to ghosts, made up of a $\lambda = 2$ fermionic bc-system and a $\lambda = 3/2$ bosonic bc-system (which we will refer to as the $\beta\gamma$-system). Generalising what happens to the bosonic string under the GO-limit, we postulate that the GO-limit for the NSR string would result in two free fermions ψ^0 and ψ^1 being replaced by a fermionic bc-system with an appropriate λ so that this new bc-system contributes a central charge of 1 to equal the contribution from ψ^0 and ψ^1. Using proposition 3.8, this condition implies that $\lambda = 1/2$.

Putting all this together, we summarise the field content of our NSR GO-string in the following table

	bc-systems	Other Fields
Matter	$(\tilde{b}, \tilde{c})$, $\lambda = 1/2$	8 free fermions ψ^i
	$(\tilde{\beta}, \tilde{\gamma})$, $\lambda = 1$	8 free bosons X^i
Ghosts	(b, c), $\lambda = 2$	
	(β, γ), $\lambda = 3/2$	

Table 5.1: A summary of all the fields in the NSR GO-string. In particular, it is natural to expect that we will focus our attention on the VOAs of the bc-systems and compute the BRST cohomology with a similar spectral sequence + KO mechanism argument. Also note that the fields of bc-systems which are part of the matter content are written with a tilde to distinguish them from the ghosts.

Apart from the change in the number of free bosons, We see that the new additions to the bosonic GO-string field content are the 8 free fermions ψ^i and the $\tilde{b}\tilde{c}$-system in the matter sector, and the $\beta\gamma$-system in the ghost sector.

5.1 Matter Sector $N = 1$ SCFT

The original NSR string worldsheet admits superconformal invariance. In other words, the symmetry algebra on the worldsheet is the $N = 1$ super-Virasoro algebra. Under the GO-limit, there is no reason for this worldsheet symmetry to change, since the it is the target space symmetry that changes. We therefore anticipate a manifestly supersymmetric formulation of the matter sector of the NSR GO-string. Of course, we know such a formulation exists for the 8 free bosons and fermions. Let us now try and build one for the $\tilde{b}\tilde{c}$ and $\tilde{\beta}\tilde{\gamma}$-systems.

We know that $\tilde{b}\tilde{c}$ and $\tilde{\beta}\tilde{\gamma}$-systems are VOAs describing CFTs. Hence, our objective is to build a primary field $\tilde{G}(z)$ from $\tilde{b}, \tilde{c}, \tilde{\beta}, \tilde{\gamma}$. As mentioned in section 3.5, doing this in a manifestly supersymmetric form means we would like to build a superfield

$$\mathbb{T}(Z) = \frac{1}{2}\tilde{G}(z) + \theta\big(T^{\tilde{b}\tilde{c}}(z) + T^{\tilde{\beta}\tilde{\gamma}}(z)\big). \tag{5.1}$$

First, we assemble the fields from the two bc-systems into superfields

$$\mathbb{B}(Z) := \tilde{b}(z) + \theta\tilde{\beta}(z) \qquad \Gamma := \tilde{\gamma}(z) + \theta\tilde{c}(z). \tag{5.2}$$

These have conformal weights $1/2$ and 0 respectively. Using (3.45), we also have

$$\begin{aligned} D\mathbb{B}(Z) &= \tilde{\beta}(z) + \theta\partial\tilde{b}(z) \qquad & D\Gamma(Z) &= \tilde{c}(z) + \theta\partial\tilde{\gamma}(z) \\ \partial\mathbb{B}(Z) &= \partial\tilde{b}(z) + \theta\partial\tilde{\beta}(z) \qquad & \partial\Gamma(Z) &= \partial\tilde{\gamma}(z) + \theta\partial\tilde{c}(z). \end{aligned} \tag{5.3}$$

The weight $3/2$ fields which we can assemble from these are

$$[\![D\mathbb{B}, D\Gamma]\!]_0, \quad [\![\mathbb{B}, \partial\Gamma]\!]_0 \quad \text{and} \quad [\![\partial\mathbb{B}, \Gamma]\!]_0.$$

Using (3.49), we write these in terms of the ordinary brackets. Note that all operator products are normal ordered but the notation has been dropped for convenience.

$$\begin{aligned} D\mathbb{B}D\Gamma(Z) &= \tilde{\beta}\tilde{c}(z) + \theta\big(\partial\tilde{b}\tilde{c}(z) - \tilde{\beta}\partial\tilde{\gamma}(z)\big) \\ \mathbb{B}\partial\Gamma(Z) &= \tilde{b}\partial\tilde{\gamma}(z) + \theta\big(\tilde{\beta}\partial\tilde{\gamma}(z) + \tilde{b}\partial\tilde{c}(z)\big) \\ \partial\mathbb{B}\Gamma(Z) &= \partial\tilde{b}\tilde{\gamma}(z) + \theta\big(\partial\tilde{\beta}\tilde{\gamma}(z) - \partial\tilde{b}\tilde{c}(z)\big). \end{aligned} \tag{5.4}$$

Thus, we expect the superfield $\mathbb{T}(Z)$ to be a linear combination of the superfields above

$$\mathbb{T}(Z) = \kappa_1 D\mathbb{B}D\Gamma(Z) + \kappa_2\mathbb{B}\partial\Gamma(Z) + \kappa_3\partial\mathbb{B}\Gamma(Z) \qquad \kappa_1, \kappa_2, \kappa_3 \in \mathbb{C}. \tag{5.5}$$

Since we know from section 3.3 that

$$T^{\tilde{b}\tilde{c}}(z) + T^{\tilde{\beta}\tilde{\gamma}}(z) = -\tilde{\beta}\partial\tilde{\gamma}(z) - \frac{1}{2}\tilde{b}\partial\tilde{c}(z) + \frac{1}{2}\partial\tilde{b}\tilde{c}(z) \tag{5.6}$$

and $\mathbb{T}(Z)$ takes the form given by (5.1), we may deduce that

$$\kappa_1 = -\kappa_2 = \frac{1}{2}.$$

Thus, we may read off the expression for $\tilde{G}$ as

$$\tilde{G} = \tilde{\beta}\tilde{c} - \tilde{b}\partial\tilde{\gamma}. \tag{5.7}$$

5.2 NSR GO-String Spectrum

This discussion will be very similar to that of the bosonic GO-string, with some care given to the incorporation of both NS and R sectors. Once again, we consider closed strings with a compact direction X^1, which gives a non-zero winding component to $\tilde{\gamma}$. First note that the BRST current is now [26]

$$j = c\big(T^{\mathfrak{M}} + \frac{1}{2}T^{bc} + T^{\beta\gamma}\big) - \gamma\tilde{G} - \gamma^2 b, \tag{5.8}$$

where the Virasoro element of the matter content and G are

$$T^{\mathfrak{M}} = T^X + T^\psi + T^{\tilde{\beta}\tilde{\gamma}} + T^{\tilde{b}\tilde{c}} \tag{5.9}$$
$$G = G^{X\psi} + \tilde{G}. \tag{5.10}$$

The BRST differential is the zero mode of the field (5.8). Having observed how the KO mechanism arose in the bosonic case, we apply the same filtration on this larger space $\mathfrak{M}$ using the same assignation of filtration degree to the modes, as summarised in table 5.2, and then extract d_0 from the full expression of d. Defining the full space of states

Mode	β_n	γ_n	b_n	c_n	$\tilde{\beta}_n$	$\tilde{\gamma}_n$	$\tilde{b}_n$	$\tilde{c}_n$	L_n^X	L_n^ψ
fdeg	-1	1	-1	1	-1	1	-1	1	0	0

Table 5.2: Summarising the assignation of filtration degrees to all the modes involved.

$$V := \mathfrak{M} \otimes \Lambda_\infty^{\bullet} \otimes V_{\beta\gamma} = \mathfrak{M}^X \otimes \mathfrak{M}^\psi \otimes V_{\tilde{\beta}\tilde{\gamma}} \otimes V_{\tilde{b}\tilde{c}} \otimes \Lambda_\infty^{\bullet} \otimes V_{\beta\gamma}, \tag{5.11}$$

the filtration F is

$$F^p V = \{v \in V \mid \operatorname{fdeg} v \geq p\}. \tag{5.12}$$

Again, by theorem A.1, we wish to compute successive pages of a spectral sequence which converges to the BRST cohomology H_d. Experience with the bosonic GO-string has taught us that the BRST differential d will split up into terms $d = \sum_{s\in\mathbb{N}^+} d_s$, where d_s raises fdeg by s, and that d_0 will come from the terms in (5.8) containing $\partial\tilde{\gamma}$. These are the $cT^{\tilde{\beta}\tilde{\gamma}}$ and $-\gamma\tilde{G}$ terms. Once again, we resort to techniques developed in Chapter 3 to write down modes of normal-ordered products of operators:

$$\begin{aligned} (cT^{\tilde{\beta}\tilde{\gamma}})_0 &= \sum_{l\leq 1} c_l L_{-l}^{\tilde{\beta}\tilde{\gamma}} + \sum_{l>1} L_{-l}^{\tilde{\beta}\tilde{\gamma}} c_l = \sum_{l\in\mathbb{Z}} c_{-l} L_l^{\tilde{\beta}\tilde{\gamma}} \\ -(\gamma\tilde{G})_0 &= -\sum_{l\leq 1/2} \gamma_l \tilde{G}_{-l} - \sum_{l>1/2} \tilde{G}_{-l}\gamma_l = -\sum_{l} \gamma_{-l}\tilde{G}_l. \end{aligned} \tag{5.13}$$

Note that the second sum runs over half-integers and integers in the NS and R sectors respectively. From proposition 3.6(*b*),

$$\begin{aligned} L_l^{\tilde{\beta}\tilde{\gamma}} &= \sum_{k\in\mathbb{Z}} k\tilde{b}_{l-k}\tilde{\gamma}_k - iwR\tilde{\beta}_l \\ \tilde{G}_l &= \sum_{k\in\mathbb{Z}} \tilde{\beta}_k \tilde{c}_{l-k} + \sum_{k\in\mathbb{Z}} k : \tilde{b}_{l-k}\tilde{\gamma}_k : - iwR\tilde{b}_l. \end{aligned} \tag{5.14}$$

The modes $L_l^{\tilde{\beta}\tilde{\gamma}}$ are labelled by integers $l \in \mathbb{Z}$ while the modes $\tilde{G}_l$ can be labelled by half-integers or integers. Nonetheless, the sum over k still runs over integers only, and the sector is determined only by the value of l. Also notice how the $\tilde{\beta}_k$ and $\tilde{\gamma}_k$ modes in $\tilde{G}_k$ always have integer labels regardless of the sectorm, which is consistent with their formulation as an integer-weighted bc-system. Now, inserting (5.14) back into (5.13) gives

$$\begin{aligned}(cT^{\tilde{\beta}\tilde{\gamma}})_0 &= \sum_{l,k\in\mathbb{Z}} k : c_l\tilde{\beta}_{l-k}\tilde{\gamma}_k : -iwR\sum_{l\in\mathbb{Z}} c_{-l}\tilde{\beta}_l \\ -(\gamma\tilde{G})_0 &= -\sum_{\substack{k\in\mathbb{Z}\\ l}} \tilde{\gamma}_{-l}\tilde{\beta}_k\tilde{c}_{l-k} - \sum_{\substack{k\in\mathbb{Z}\\ l}} k\tilde{\gamma}_{-l}\tilde{\beta}_{l-k}\tilde{\gamma}_k + iwR\sum_l \gamma_{-l}\tilde{b}_l.\end{aligned} \tag{5.15}$$

Referring to table 5.2, we construct d_0

$$d_0 = \sum_{m\in\mathbb{Z}} c_{-m}\tilde{\beta}_m - \sum_n \gamma_{-n}\tilde{b}_n, \tag{5.16}$$

where the sum over n runs over half-integers and integers in the NS and R sectors respectively.

To implement the KO mechanism, we need to define the vacuum

$$|\mathrm{vac}\rangle = |\mathrm{vac}_1\rangle \otimes |\mathrm{vac}_2\rangle\,, \tag{5.17}$$

where $|\mathrm{vac}_1\rangle = |p_1\rangle_{\tilde{\beta}\tilde{\gamma}} \otimes |q_1\rangle_{bc}$ and $|\mathrm{vac}_2\rangle = |p_2\rangle_{\beta\gamma} \otimes |q_2\rangle_{\tilde{b}\tilde{c}}$, and $p_1, q_1 \in \mathbb{Z}$ while p_2, q_2 are either half-integers or integers depending on whether we are in the NS or R sector. Just as in the bosonic case, we have an infinite number of choices for $|\mathrm{vac}_1\rangle_{M\in\mathbb{N}^+} = |-M\rangle_{\tilde{\beta}\tilde{\gamma}} \otimes |M+1\rangle_{bc}$, which we obtained by requiring that $\tilde{\beta}_0\,|p_1\rangle_{\tilde{\beta}\tilde{\gamma}} \neq 0$ and the "matching" condition on $|p_1\rangle_{\tilde{\beta}\tilde{\gamma}}$ and $|q_1\rangle_{bc}$. We demand a similar matching condition for $|p_2\rangle_{\beta\gamma}$ and $|q_2\rangle_{\tilde{b}\tilde{c}}$, but we do not need the analogous condition of β_0 to annihilate $|p_2\rangle_{\beta\gamma}$, since non-trivial solutions for $L_0^{\mathrm{tot}} = 0$ does not impose this. The vacuum conditions (3.26) with $\epsilon = -1$ and $\lambda = 3/2$ are

$$\begin{aligned}\beta_n\,|p_2\rangle &= 0 \quad n > -p - \frac{3}{2} \qquad \tilde{b}_n\,|q_2\rangle = 0 \qquad n > q - \frac{1}{2} \\ \gamma_n\,|p_2\rangle &= 0 \quad n > p + \frac{3}{2} \qquad \tilde{c}_n\,|q_2\rangle = 0 \quad n \geq -q + \frac{1}{2}.\end{aligned} \tag{5.18}$$

When p_2 and q_2 are half-integers, we are in the NS sector, so the modes are also labelled by half-integers n. Likewise, when we are in the R-sector and $p_2, q_2 \in \mathbb{Z}$, the modes are labelled $n \in \mathbb{Z}$. Hence $n - p_2$ and $n - q_2$ are always integers, and we may write (5.18) as

$$\begin{aligned}\beta_n\,|p_2\rangle &= 0 \quad n > -p - 2 \qquad \tilde{b}_n\,|q_2\rangle = 0 \qquad n > q - 1 \\ \gamma_n\,|p_2\rangle &= 0 \quad n > p + 2 \qquad \tilde{c}_n\,|q_2\rangle = 0 \quad n \geq -q + 1.\end{aligned} \tag{5.19}$$

The matching condition demands that if β_n (resp. γ_n) is an annihilator for some n, so is $\tilde{b}_n$ (resp. $\tilde{c}_n$). This relates p_2 and q_2 by

$$-p_2 - 2 = q_2 - 1 \implies q_2 = -p_2 - 1. \tag{5.20}$$

Letting $N = -p_2$, we see that we once again have an infinite number of choices for $|\mathrm{vac}_2\rangle$, labelled by N,

$$|\mathrm{vac}_2\rangle_N = |-N\rangle_{\beta\gamma} \otimes |N-1\rangle_{\tilde{b}\tilde{c}} \qquad N \in \begin{cases} \mathbb{Z}+\frac{1}{2} & \text{NS sector.} \\ \mathbb{Z} & \text{R sector.} \end{cases} \tag{5.21}$$

Having constructed $|\mathrm{vac}_2\rangle$, we may assign creation and annihilation operators to each mode as follows:

$$\begin{array}{llll} a^\dagger(\tilde{\beta}_m) = \tilde{\beta}_{-m} & m \geq 1-M & a^\dagger(\tilde{\gamma}_m) = -\tilde{\gamma}_{-m} & m \geq M \\ a(\tilde{\beta}_m) = \tilde{\gamma}_m & m \geq 1-M & a(\tilde{\gamma}_m) = \tilde{\beta}_m & m \geq M \\ a^\dagger(b_m) = b_{-m} & m \geq 1-M & a^\dagger(c_m) = c_{-m} & m \geq M \\ a(b_m) = c_m & m \geq 1-M & a(c_m) = b_m & m \geq M \\ a^\dagger(\beta_n) = \beta_{-n} & n \geq 2-N & a^\dagger(\gamma_n) = -\gamma_{-n} & n \geq N-1 \\ a(\beta_n) = \gamma_n & n \geq 2-N & a(\gamma_n) = \beta_n & n \geq N-1 \\ a^\dagger(\tilde{b}_n) = \tilde{b}_{-n} & n \geq 2-N & a^\dagger(\tilde{c}_n) = \tilde{c}_{-n} & n \geq N-1 \\ a(\tilde{b}_n) = \tilde{c}_n & n \geq 2-N & a(\tilde{c}_n) = \tilde{b}_n & n \geq N-1 \end{array} \tag{5.22}$$

Applying (5.22) to (5.16) gives

$$d_0 = \sum_{m\geq M} a^\dagger(c_m)a(\tilde{\gamma}_m) + \sum_{n\geq 1-M} a^\dagger(\tilde{\beta}_m)a(b_m) + \sum_{n\geq N-1} a^\dagger(\gamma_n)a(\tilde{c}_n) - \sum_{n\geq 2-N} a^\dagger(\tilde{b}_n)a(\beta_n). \tag{5.23}$$

From the above, we infer the expression for K

$$K = \sum_{m\geq M} a^\dagger(\tilde{\gamma}_m)a(c_m) + \sum_{n\geq 1-M} a^\dagger(b_m)a(\tilde{\beta}_m) + \sum_{n\geq N-1} aa^\dagger(\tilde{c}_n)(\gamma_n) - \sum_{n\geq 2-N} a^\dagger(\beta_n)a(\tilde{b}_n), \tag{5.24}$$

using which we construct the number operator $N_{tot} = d_0K + Kd_0$. Through an identical but lengthier computation on monomials that span the space $V_{\tilde{\beta}\tilde{\gamma}} \otimes V_{\tilde{b}\tilde{c}} \otimes \Lambda^{\bullet}_{\infty} \otimes V_{\beta\gamma}$, one can verify the action of N_{tot}. Thus, by the KO mechanism, we once again arrive at the conclusion that the d_0-cohomology consists of only one non-trivial class, represented by $|\mathrm{vac}\rangle$. By the same arguments we made for the bosonic case, $d_1 \equiv 0$ on H_{d_0} and so the spectral sequence collapses, leaving us with the BRST cohomology

$$H_d \cong H_{d_0} \cong \mathbb{C}\,|\mathrm{vac}\rangle = \mathbb{C}\left(|-M\rangle_{\tilde{\beta}\tilde{\gamma}} \otimes |M+1\rangle_{bc} \otimes |N\rangle_{\beta\gamma} \otimes |N-1\rangle_{\tilde{b}\tilde{c}}\right), \tag{5.25}$$

where M and N take the values summarised in the table below:

	NS	R
M	$\mathbb{N}^+$	$\mathbb{N}^+$
N	$\mathbb{Z}+\frac{1}{2}$	$\mathbb{Z}$

Hence, in either sector, we have an infinite number of pictures labelled by a pair (M, N) that take the above values. Every picture gives isomorphic cohomology.

Chapter 6

Conclusions and Next Steps

We have developed an algebraic framework that allows us to set up computations needed for obtaining string theory spectra. This framework was effective in the case of the Gomis-Ooguri string, as seen by its ability to reproduce the original spectrum, while also being able to address the caveat of having infinite pictures. We were then able to extend the original Gomis-Ooguri limit to NSR strings by hypothesising that two fermionic fields that entered the relativistic action would also be replaced by a fermionic bc-system of suitable weight (which was computed to be $\lambda = 1/2$). This allowed us to apply similar algebraic methods to compute the spectrum of the NSR string under this limit. We obtained what one would naively expect; an extra label N, which is either a half-integer or integer depending on whether we are in the NS or R sector, which labels the infinite pictures resulting from an infinite number of choices for the $\tilde{\beta}\tilde{\gamma}$ and $\beta\gamma$ vacua.

Hopefully, this report has served its primary function of providing a foundation to explore other interesting areas of non-relativistic string theory. Immediate next steps (inspired by a brief discussion with Jelle Hartong), could involve looking at bosonic and NSR versions of open strings to better understand non-relativistic versions of D-branes. Another possibility is to study the non-relativistic limit[§] of the Green-Schwarz superstring (see [28] for a discussion of this for $D = 4$ and $N = 1$). The potential implications of these and other possible paths forward need further discussion.

[§]Perhaps this is not a precise statement; based on [27], one could take such a limit in two ways

Appendix A

Spectral Sequences Review

This chapter provides a gentle introduction to spectral sequences and some of their key features. It is based on chapter 2 of this book [29] by McCleary. The discussion here makes use of R-modules for a general ring R, but for the purpose of string theory, $R = \mathcal{U}(\mathfrak{g})$, where $\mathfrak{g}$ is the Virasoro algebra. It should be mentioned that this exposition is by no means a thorough one; it is only intended to lay the groundwork for the application of spectral sequences to the computation of string theory spectra.

A.1 Basic Setup

Definition 16: A **differential bigraded module** over a ring R is a collection of R-modules $\{E^{p,q}\}_{p,q\in\mathbb{Z}}$ with an R-linear differential $d : E^{\bullet,\bullet} \to E^{\bullet,\bullet}$ of bidegree $(r, 1-r)$ for some $r \in \mathbb{Z}$ such that $d^2 = d \circ d = 0$.

Definition 17: A (cohomological) **spectral sequence** is a collection of differential bigraded modules $\{E_r^{\bullet,\bullet}, d_r\}$ where $r = 0, 1, 2, \ldots$ where for all p, q, r,

$$E_{r+1}^{p,q} \cong H^{p,q}(E^{\bullet,\bullet}, d_r) := \frac{\ker\left(d_r : E_r^{p,q} \longrightarrow E_r^{p+r,q+1-r}\right)}{\operatorname{im}\left(d_r : E_r^{p-r,q-1+r} \longrightarrow E_r^{p,q}\right)}. \tag{A.1}$$

The collection of R-modules $E_r^{\bullet,\bullet}$ is known as the r^{th} **page** of the spectral sequence. Figure A.1 illustrates the relationship between pages $r = 0$ and $r = 1$.

We would like identify the end goal of computations with spectral sequences. The most intuitive thing to think about would be to keep turning the pages of a spectral sequence until one reaches the $r \to \infty$ page of the spectral sequence. Let us now define this more precisely by elucidating its origins. We suppress the (p, q) indices for clarity and start with page 2 of the spectral sequence for concreteness. Define $Z_2 := \ker d_2$ and $B_2 = \operatorname{im} d_2$. Then

$$d_2 \circ d_2 = 0 \implies B_2 \subset Z_2 \subset E_2.$$

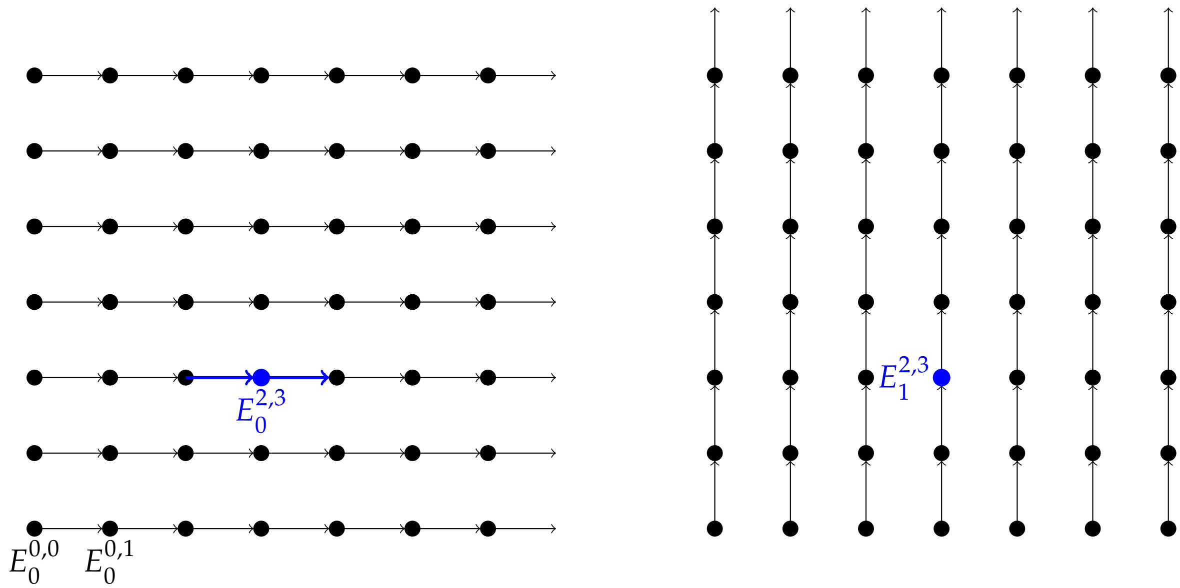

Figure A.1: The zeroth page (left) and the first page (right) of a spectral sequence. The grading p increases along the y-axis while q increases along the x-axis and the arrows depict the action of d_r. The cohomology at $E_0^{2,3}$ determines $E_1^{2,3}$ as highlighted in blue. However, note that it does *not* determine d_1.

Now define $\bar{Z}_3 := \ker d_3$ and $\bar{B}_3 := \operatorname{im} d_3$. Since these are submodules of $E_3 = Z_2 / B_2$, there exist $B_3 \subset Z_3 \subset E_3$ such that $\bar{Z}_3 = Z_3 / B_2$ and $\bar{B}_3 = B_3 / B_2$. Consequently, $E_4 = \bar{Z}_3 / \bar{B}_3 = Z_3 / B_3$ and we have the tower of inclusions

$$B_2 \subset B_3 \subset Z_3 \subset Z_2.$$

Iterating, the spectral sequence can be presented as an infinite tower of inclusions

$$B_2 \subset B_3 \ldots B_n \subset B_{n+1} \ldots \quad \ldots Z_n \subset Z_{n-1} \subset \cdots \subset Z_3 \subset Z_2$$

with $E_{n+1} = Z_n / B_n$, and differentials

$$d_{n+1} \colon Z_n / B_n \to Z_n / B_n \qquad \ker d_{n+1} = Z_{n+1} / B_n \qquad \operatorname{im} d_{n+1} = B_{n+1} / B_n .$$

This gives rise to a short exact sequence for each n

$$0 \longrightarrow Z_{n+1} / B_n \hookrightarrow Z_n / B_n \xrightarrow{d_{n+1}} B_{n+1} / B_n \longrightarrow 0. \tag{A.2}$$

An element in E_2 is said to **survive the r^{th} stage** if it lies in Z_r. Likewise, an element in E_2 is said to be a **boundary by the r^{th} stage** if it lies in B_r. We may then define

$$Z_\infty := \bigcap_n Z_n \qquad B_\infty := \bigcup_n B_n \tag{A.3}$$

as the submodules of E_2 of elements that **survive forever** and are **eventually bound** respectively. Looking back at the infinite tower of inclusions, we indeed have $B_\infty \subset Z_\infty$ and thus, we may sensibly define what we sought:

$$E_\infty := {}^{Z_\infty}\big/_{B_\infty}\,. \tag{A.4}$$

This is the bigraded module that one obtains from the calculation of the infinite sequence of successive cohomologies. In some cases, this calculation truncates at some finite stage.

Definition 18: A spectral sequence is said to **collapse at the N^{th} term** if $d_{r\geq N} = 0$.

From (A.2), $d_{r=N} = 0 \implies Z_N = Z_{N-1}$, since the kernel of the zero map d_N is simply ${}^{Z_{N-1}}\big/_{B_{N-1}}$ which should equal the image ${}^{Z_N}\big/_{B_{N-1}}$ of the includion map. Likewise,

$$\ker\left({}^{B_N}\big/_{B_{N-1}} \longrightarrow 0\right) = {}^{B_N}\big/_{B_{N-1}} = \operatorname{im} d_N = 0 \iff B_N = B_{N+1}.$$

The tower of inclusions then becomes

$$B_2 \subset B_3 \subset \cdots \subset B_{N-1} = B_N = \cdots = B_\infty \subset Z_\infty = \cdots = Z_N = Z_{N-1} \subset \cdots \subset Z_3 \subset Z_2,$$

thereby reducing the computation to a finite one as promised. It is this collapse that we will exploit when using spectral sequences to compute string theory spectra. *Define convergence of a spectral sequnce.*

A.2 Spectral Sequences from Filtrations

One way a spectral sequence can arise is through a filtration. Suppose $\{A^\bullet = \bigoplus_{n\in\mathbb{Z}} A^n, d\}$ is a differential graded R-module with $d\colon A^n \to A^{n+1}$. Then we apply a decreasing filtration (commonly the case for cohomological spectral sequences)

$$F : \cdots \subseteq F^{p+1}A \subseteq F^pA \subseteq F^{p-1}A \subseteq \ldots$$

such that the differential respects the filtration (so $d\colon F^pA \to F^pA$). We may also define the filtration at each graded level via $F^pA^n := F^pA \cap A^n$. Then $d\colon F^pA^n \to F^pA^{n+1}$.

Definition 19: For any decreasing filtration F on a differential graded module $\{A^\bullet, d\}$, the **associated bigraded module** is given by

$$E_0^{p,q}(A^\bullet, F) := {}^{F^pA^{p+q}}\big/_{F^{p+1}A^{p+q}}\,. \tag{A.5}$$

For our purposes, the bigraded module associated to a filtration F will be how the zeroth page of a spectral sequence emerges (hence the compatible notation), so let us proceed under the assumption that such a spectral sequence exists, with d_0 the part of d that leaves the filtration degree unchanged.

Now consider figure A.3. There are two natural operations that we can perform; taking

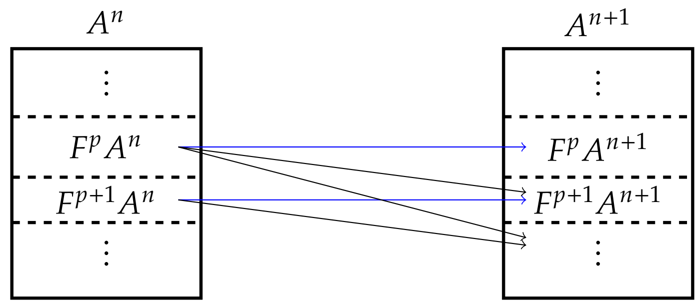

Figure A.2: The arrows depict the possible actions of d. The filtration degree of any element in A^n is never raised by d. The blue arrows depict d_0.

$$\begin{tikzcd}
 & \vdots & \vdots & \vdots & \\
\dots \arrow[r, "d"] & F^{p-1}A^{n-1} \arrow[r, "d"] \arrow[u, hook] & F^{p-1}A^n \arrow[r, "d"] \arrow[u, hook] & F^{p-1}A^{n+1} \arrow[r, "d"] \arrow[u, hook] & \dots \\
\dots \arrow[r, "d"] & F^pA^{n-1} \arrow[r, "d"] \arrow[u, hook] & F^pA^n \arrow[r, "d"] \arrow[u, hook] & F^pA^{n+1} \arrow[r, "d"] \arrow[u, hook] & \dots \\
\dots \arrow[r, "d"] & F^{p+1}A^{n-1} \arrow[r, "d"] \arrow[u, hook] & F^{p+1}A^n \arrow[r, "d"] \arrow[u, hook] & F^{p+1}A^{n+1} \arrow[r, "d"] \arrow[u, hook] & \dots \\
 & \vdots \arrow[u, hook] & \vdots \arrow[u, hook] & \vdots \arrow[u, hook] &
\end{tikzcd}$$

Figure A.3: The structure of the filtered differential graded module with the action of d.

quotients along the inclusions and cohomologies along d. If we do the latter first, we observe that there is an induced filtration (which we will also denote by F) on the cohomology $H^\bullet(A,d)$.

$$F^pH^n(A,d) := \frac{\ker\big(d\colon F^pA^n \longrightarrow F^pA^{n+1}\big)}{\operatorname{im}\big(d\colon F^pA^{n-1} \longrightarrow F^pA^n\big)} =: H^n(F^pA,d). \tag{A.6}$$

This amounts to taking the cohomology at each filtration degree. As a result $\{H^\bullet, d, F\}$ is also a filtered differential module, from which we may construct the associated graded module $E_0^{p,q}\big(H^\bullet(A,d),F\big)$. If instead we do the former (take quotients) first, we obtain the chain complex $\{E_0^{\bullet,\bullet}(A^\bullet,F), d_0\}$:

$$\dots \xrightarrow{d_0} \frac{F^pA^{n-1}}{F^{p+1}A^{n-1}} \xrightarrow{d_0} \frac{F^pA^n}{F^{p+1}A^n} \xrightarrow{d_0} \frac{F^pA^{n+1}}{F^{p+1}A^{n+1}} \xrightarrow{d_0} \dots \tag{A.7}$$

The differential is now d_0 instead of d, because d_0 is the only part of d that is non-vanishing on the quotients since the rest of d would raise the fitlration degree. Letting $n = p + q$ for clarity, we obtain

$$H^{p+q}\left({F^pA}\big/{F^{p+1}A}\,, d_0\right) = \frac{\ker\left(d\colon\; {F^pA^{p+q}}\big/{F^{p+1}A^{p+q}} \longrightarrow {F^pA^{p+q+1}}\big/{F^{p+1}A^{p+q+1}}\right)}{\operatorname{im}\left(d\colon\; {F^pA^{p+q-1}}\big/{F^{p+1}A^{p+q-1}} \longrightarrow {F^pA^{p+q}}\big/{F^{p+1}A^{p+q}}\right)}. \tag{A.8}$$

The RHS is precisely $H^{p,q}\big(E_0^{\bullet,\bullet}(A,F),d_0\big) \cong E_1^{p,q}$, which would be page 1 of our spectral sequence. Hence, we have obtained two different objects, $E_0^{p,q}\big(H^\bullet(A,d),F\big)$ and $H^{p,q}\big(E_0^{\bullet,\bullet}(A,F),d_0\big)$, and the following equivalent statements by taking quotients and cohomologies in different orders:

1. The associated bigraded module $E_0^{p,q}(A^\bullet,F)$ is the zeroth page of a spectral sequence.
2. The first page of a spectral sequence $E_1^{p,q}$ is isomorphic to $H^{p+q}\left(F^pA/F^{p+1}A, d_0\right)$.

Recall that we had constructed E_∞ by laying out the spectral sequence as an infinite tower of inclusions. We now introduce the notion of convergence.

Definition 20: Let $\mathcal{A}^\bullet$ be a graded R-module. A spectral sequence **converges** to $\mathcal{A}^\bullet$ if there exists a filtration $\mathcal{F}$ on $\mathcal{A}^\bullet$ such that

$$E_\infty^{p,q} \cong E_0^{p,q}(\mathcal{A}^\bullet,\mathcal{F}). \tag{A.9}$$

The interplay between $E_\infty^{p,q}$, $E_0^{p,q}\big(H^\bullet(A,d),F\big)$ and $H^{p,q}\big(E_0^{\bullet,\bullet}(A,F),d_0\big)$ is encapsulated in the following theorem.

Theorem A.1: *Each filtered differential graded module $\{A^\bullet,d,F\}$ determines a spectral sequence $\{E_r^{\bullet,\bullet},d_r\}_{r\in\mathbb{N}^+}$ with*

$$E_1^{p,q} \cong H^{p+q}\left({F^pA}\big/{F^{p+1}A}\,, d_0\right) = H^{p,q}\big(E_0^{\bullet,\bullet}(A,F),d_0\big). \tag{A.10}$$

Suppose also that F is a ***bounded filtration****, meaning $\exists s(n), t(n) \in \mathbb{Z}$ for every n such that*

$$0 = F^{s(n)}A^n \subseteq F^{s(n)-1}A^n \subseteq \cdots \subseteq F^{t(n)+1}A^n \subset F^{t(n)}A^n = A^n. \tag{A.11}$$

Then the spectral sequences converges to $H^\bullet(A,d)$:

$$E_\infty^{p,q} \cong E_0^{p,q}\big(H^\bullet(A,d),F\big) = {F^pH^{p+q}(A,d)}\big/{F^{p+1}H^{p+q}(A,d)}\,. \tag{A.12}$$

This is the key theorem from spectral sequences which was utilised in [8] to prove the vanishing theorem in semi-infinite cohomology. A detailed proof is given in chapter 2.2 of McCleary's book [29]. However, examination of the proof will quickly reveal that boundedness of the filtration is too strong of a condition. Boundedness will guarantee the desired convergence of the spectral sequence in general, but if we have explicit knowledge of the filtration used in a specific example, we only need it to satisfy weaker conditions.

Definition 21: Let $\{A^\bullet = \bigoplus_{n\in\mathbb{Z}} A^n, d\}$ be a differential graded R-module and let F be a stable filtration (preserved by d). F is **exhaustive** if $\bigcup_p F^p A = A^\bullet$ and **weakly convergent** if $\bigcap_p F^p A = 0$.

Theorem A.1 then holds for an exhaustive and weakly convergent filtration rather than a bounded one. We will make use of this version of the theorem when computing the BRST cohomology of non-relativistic strings.

Appendix B

Proofs and Calculations

B.1 Semi-Infinite Cohomology

Proof of Proposition 2.3: We will perform calculations on monomials ω and the argument extends to all semi-infinite forms by $\mathbb{C}$-linearity.

$$\begin{aligned}
[\rho(x), \varepsilon(y')]e'_{i_1} \wedge e'_{i_2} \wedge \ldots &= \mathrm{ad}'_x\, y' \wedge e'_{i_1} \wedge e'_{i_2} \wedge \cdots + \sum_{k\geq 1} y' \wedge e'_{i_1} \wedge e'_{i_2} \wedge \cdots \wedge \mathrm{ad}'_x\, e'_{i_k} \wedge \ldots \\
&\quad - y' \wedge \sum_{k\geq 1} e'_{i_1} \wedge e'_{i_2} \wedge \cdots \wedge \mathrm{ad}'_x\, e'_{i_k} \wedge \ldots \\
&= \mathrm{ad}'_x\, y' \wedge e'_{i_1} \wedge e'_{i_2} \wedge \ldots \\
&= \varepsilon(\mathrm{ad}'_x\, y')e'_{i_1} \wedge e'_{i_2} \wedge \ldots
\end{aligned}$$

$$\begin{aligned}
[\rho(x), \iota(y)]e'_{i_1} \wedge e'_{i_2} \wedge \ldots &= \rho(x) \sum_{k\geq 1}(-1)^{k-1}\langle y, e'_{i_k}\rangle e'_{i_1} \wedge e'_{i_2} \wedge \cdots \wedge \widehat{e'_{i_k}} \wedge \ldots \\
&\quad - \iota(y) \sum_{k\geq 1} e'_{i_1} \wedge e'_{i_2} \wedge \cdots \wedge \mathrm{ad}'_x\, e'_{i_k} \wedge \ldots \\
&= \sum_{k\geq 1}(-1)^{k-1} - \langle y, \mathrm{ad}'_x\, e'_{i_k}\rangle e'_{i_1} \wedge e'_{i_2} \wedge \cdots \wedge \widehat{\mathrm{ad}'_x\, e'_{i_k}} \wedge \ldots \\
&= \sum_{k\geq 1}(-1)^{k-1}\langle \mathrm{ad}_x\, y, e'_{i_k}\rangle e'_{i_1} \wedge e'_{i_2} \wedge \cdots \wedge \widehat{e'_{i_k}} \wedge \ldots \\
&= \iota(\mathrm{ad}_x\, y)e'_{i_1} \wedge e'_{i_2} \wedge \ldots
\end{aligned}$$

■

Proof Of Proposition 2.6: Akman has shown in [15] that proving proposition 2.6 is equivalent (in our case) to proving that

$$\theta(x) = \rho(x) + \pi(x) = d\iota(x) + \iota(x)d. \tag{B.1}$$

It suffices to show that (B.1) is true for a basis element $x = e_r$ since we may write any $x \in\geq$ as $x = \sum_r a_r e_r$, where only a finite number of a_r are non-zero by construction of $\mathfrak{g}$ as a direct sum. First explicitly write out the normal ordered part of d

$$\sum_{i<j} :\iota([e_i, e_j])\varepsilon(e'_j)\varepsilon(e'_i) := \sum_{\substack{i+j\leq i_0 \\ i<j}} \iota([e_i, e_j])\varepsilon(e'_j)\varepsilon(e'_i) + \sum_{\substack{i+j>i_0 \\ i<j}} \varepsilon(e'_j)\varepsilon(e'_i)\iota([e_i, e_j]) \tag{B.2}$$

Then we have

$$\begin{aligned} d\iota(e_r) + \iota(e_r)d = &\sum_{\substack{i+j\leq i_0 \\ i<j}} \Big(\iota([e_i, e_j])\varepsilon(e'_j)\varepsilon(e'_i)\iota(e_r) + \iota(e_r)\iota([e_i, e_j])\varepsilon(e'_j)\varepsilon(e'_i)\Big) \\ &+ \sum_{\substack{i+j>i_0 \\ i<j}} \Big(\varepsilon(e'_j)\varepsilon(e'_i)\iota([e_i, e_j])\iota(e_r) + \iota(e_r)\varepsilon(e'_j)\varepsilon(e'_i)\iota([e_i, e_j])\Big) \\ &+ \sum_{i\in\mathbb{Z}} \pi(e_i)\big(\varepsilon(e'_i)\iota(e_r) + \iota(e_r)\varepsilon(e'_i)\big). \end{aligned} \tag{B.3}$$

The last line is equal to $\pi(e_r)$ by proposition 2.1. For the sums, we may manipulate the terms to be written in the same order using proposition 2.1. For example,

$$\begin{aligned} &\iota([e_i, e_j])\varepsilon(e'_j)\varepsilon(e'_i)\iota(e_r) + \iota(e_r)\iota([e_i, e_j])\varepsilon(e'_j)\varepsilon(e'_i) \\ =&\iota([e_i, e_j])\varepsilon(e'_j)\big(\delta_{ir} - \iota(e_r)\varepsilon(e'_i)\big) - \iota([e_i, e_j])\big(\delta_{jr} - \varepsilon(e'_j)\iota(e_r)\big)\varepsilon(e'_i) \\ =&\iota([e_i, e_j])\varepsilon(e'_j)\delta_{ir} - \iota([e_i, e_j])\varepsilon(e'_i)\delta_{jr}. \end{aligned} \tag{B.4}$$

We therefore have

$$\begin{aligned} d\iota(e_r) + \iota(e_r)d = \pi(e_r) &+ \sum_{\substack{i+j\leq i_0 \\ i<j}} \Big(\iota([e_i, e_j])\varepsilon(e'_j)\delta_{ir} - \iota([e_i, e_j])\varepsilon(e'_i)\delta_{jr}\Big) \\ &+ \sum_{\substack{i+j>i_0 \\ i<j}} \Big(-\varepsilon(e'_j)\iota([e_i, e_j])\delta_{ir} + \varepsilon(e'_i)\iota([e_i, e_j])\delta_{jr}\Big). \end{aligned} \tag{B.5}$$

Relabelling $i \leftrightarrow j$ in the second term of each sum gives

$$\begin{aligned} d\iota(e_r) + \iota(e_r)d &= \pi(e_r) + \sum_{\substack{i+j\leq i_0 \\ i,j\in\mathbb{Z}}} \Big(\iota([e_i, e_j])\varepsilon(e'_j)\delta_{ir}\Big) - \sum_{\substack{i+j>i_0 \\ i,j\in\mathbb{Z}}} \Big(\varepsilon(e'_j)\iota([e_i, e_j])\delta_{ir}\Big) \\ &= \pi(e_r) + \sum_{j\in\mathbb{Z}} :\iota(\mathrm{ad}_{e_r} e_j)\varepsilon(e'_j): \\ &= \pi(e_r) + \rho(e_r). \end{aligned} \tag{B.6}$$

Making use of (2.15) finishes the proof in the last equality.

$\blacksquare$

Proof Of Proposition 2.7: We start by splitting $\rho(x)$ as

$$\rho(x) = S(x) + \langle \beta, x \rangle, \tag{B.7}$$

where $S(x)$ is the normal ordered part in (2.13). Then

$$S(\sigma(x)) = \sum_{i\in\mathbb{Z}} :\varepsilon(\mathrm{ad}'_{\sigma(x)} e'_i)\iota(e_i): \tag{B.8}$$

For all $y \in \mathfrak{g}$,

$$\begin{aligned}
\langle \mathrm{ad}'_{\sigma(x)} e'i, y\rangle &= -\langle e'i, \mathrm{ad}_{\sigma(x)} y\rangle && \text{(by ad-invariance of } \langle -,-\rangle)\\
&= -\langle e'_i, [\sigma(x), y]\rangle && \text{(by definition of ad)}\\
&= -\langle e'_i, [\sigma(x), \sigma(\sigma(y))] && (\sigma \text{ is involutive})\\
&= -\langle e'_i, \sigma([x, \sigma(y)])\rangle && (\sigma \text{ is an automorphism})\\
&= -\overline{\langle \sigma(e'_i), [x, \sigma(y)]\rangle} && \text{(from (2.22))}\\
&= \overline{\langle \mathrm{ad}'_x \sigma(e'_i), \sigma(y)\rangle} && \text{(by ad-invariance of } \langle -,-\rangle)\\
&= \overline{\langle \sigma^2(\mathrm{ad}'_x \sigma(e'_i)), \sigma(y)\rangle} && (\sigma \text{ is involutive})\\
&= \langle \sigma(\mathrm{ad}'_x \sigma(e'_i)), y\rangle && \text{(from (2.22))}
\end{aligned}$$

Therefore,

$$\mathrm{ad}'_{\sigma(x)} e'i = \sigma(\mathrm{ad}'_x \sigma(e'_i)). \tag{B.9}$$

Also, (2.23) implies that $\iota(e_i) = -\iota(\sigma(x))^\dagger$. Thus,

$$\begin{aligned}
S(\sigma(x)) &= \sum_{i\in\mathbb{Z}} :\varepsilon\big(\sigma(\mathrm{ad}'_x \sigma(e'_i))\big)\iota(e_i): \\
&= \sum_{i\in\mathbb{Z}} :\varepsilon\big(\mathrm{ad}'_x \sigma(e'_i)\big)^\dagger \iota\big(\sigma(e_i)\big)^\dagger : \\
&= \sum_{i\in\mathbb{Z}} :\varepsilon\big(\mathrm{ad}'_x \sigma(e'_i)\big)^\dagger \iota\big(\sigma(e_i)\big)^\dagger : \\
&= \left(\sum_{i\in\mathbb{Z}} :\iota\big(\sigma(e_i)\big)\varepsilon\big(\mathrm{ad}'_x \sigma(e'_i)\big):\right)^\dagger \\
&= -\left(\sum_{i\in\mathbb{Z}} :\varepsilon\big(\mathrm{ad}'_x \sigma(e'_i)\big)\iota\big(\sigma(e_i)\big):\right)^\dagger
\end{aligned} \tag{B.10}$$

Now observe that the last summation is actually just $S(x)$. This is because σ, by virtue of being an automorphism, defines a new basis on $\mathfrak{g}$ via $f_i := \sigma(e_i)$ and its dual on $\mathfrak{g}'$ via $f'_i = \sigma(e'_i)$, since

$$\langle f'_i, f_j\rangle = \langle \sigma(e'_i), \sigma(e_j)\rangle = \overline{\langle e'_i, e_j\rangle} = \delta_{ij}.$$

Thus,

$$S(\sigma(x)) = -\left(\sum_{i\in\mathbb{Z}} :\varepsilon\left(\mathrm{ad}'_x f'_i\right)\iota\left(f_i\right)\right)^{\dagger} := -S(x)^{\dagger}. \tag{B.11}$$

Now

$$\overline{\langle\beta,[x,y]\rangle} = \langle\sigma(\beta),\sigma([x,y])\rangle.$$

On the other hand,

$$[\rho(x),\rho(y)] = [S(x),S(y)] = \rho([x,y]) = S([x,y]) + \langle\beta,[x,y]\rangle,$$

which tells us that

$$\langle\beta,[x,y]\rangle = [S(x),S(y)] - S([x,y]).$$

Then

$$\begin{aligned}\overline{\langle\beta,[x,y]\rangle} &= \langle\beta,[x,y]\rangle^{\dagger}\\ &= [S(x),S(y)]^{\dagger} - S([x,y])^{\dagger}\\ &= [S(y)^{\dagger},S(x)^{\dagger}] + S(\sigma([x,y]))\\ &= -[S(\sigma(x)),S(\sigma(y)] + S([\sigma(x),\sigma(y)])\\ &= -\langle\beta,[\sigma(x),\sigma(y)]\rangle\\ &= -\langle\beta,\sigma([x,y])\rangle\end{aligned} \tag{B.12}$$

Equating the two expressions for $\overline{\langle\beta,[x,y]\rangle}$ gives

$$\langle\sigma(\beta),\sigma([x,y])\rangle = -\langle\beta,\sigma([x,y])\rangle \qquad \forall x,y\in\mathfrak{g}. \tag{B.13}$$

Thus, $\sigma(\beta) = -\beta$.

■

Proof Of Proposition 2.8: From (B.1)

$$\theta(x)^{\dagger} = \left(d\iota(x) + \iota(x)d\right)^{\dagger} \tag{B.14}$$

The LHS is equal to $-\theta(x)$ while the RHS gives $-\iota(x)d^{\dagger} - d^{\dagger}\iota(x)$, where we have used (2.23) and $(\phi_1\phi_2)^{\dagger} = \phi_2^{\dagger}\phi_1^{\dagger}$ for any $\phi_1,\phi_2\in\mathrm{End}(\mathfrak{M}\otimes\Lambda^{\bullet}_{\infty})$. Demanding that the LHS equals the RHS forces $d^{\dagger} = d$.

■

Proof Of Proposition 2.11: This proof is shown in the proof of proposition 3.9 as it is more fitting in that context. After all, this result is specific to the Virasoro algebra, the semi-infinite cohomology of which occurs as a VOA, so it is more fitting to make use VOA techniques to prove these results.

B.2 Vertex Operator Algebras

Proof of Proposition 3.1: From the definition of ∂,

$$(\partial A)(z)B(W) := \frac{d}{dz}A(z)B(w) = \frac{d}{dz}\sum_{n\ll\infty}\frac{[A,B]_n(w)}{(z-w)^n} = \sum_{n\ll\infty}\frac{-n[A,B]_n(w)}{(z-w)^{n+1}}.$$

But $(\partial A)(z)B(w)$ itself admits an OPE

$$(\partial A)(z)B(w) = \sum_{n\ll\infty}\frac{[\partial A,B]_n(w)}{(z-w)^n}.$$

Equating equal powers of $z-w$ gives

$$[\partial A,B]_{n+1} = -n[A,B]_n \iff [\partial A,B]_n = -(n-1)[A,B]_{n-1}.$$

This proves (a). (b) can be proved in a similar manner. On one hand,

$$A(z)\partial B(w) = \sum_{n\ll\infty}\frac{[A,\partial B]_n}{(z-w)^n},$$

while on the other hand,

$$A(z)\partial B(w) := \frac{d}{dw}\sum_{n\ll\infty}\frac{[\partial A,B]_n(w)}{(z-w)^n} = \sum_{n\ll\infty}\frac{\partial[A,B]_n(w)}{(z-w)^n} + \sum_{n\ll\infty}\frac{n[A,B]_n(w)}{(z-w)^{n+1}}$$

Relabelling the summation index and equating equal powers of $z-w$ once again gives the desired result. (c) follows immediately from (a) and (b):

$$\begin{aligned}\partial[A,B]_n &= [A,\partial B] - (n-1)[A,B]_{n-1} && \text{(from } (b)\text{)}\\ &= [\partial A,B]_n + [A,\partial B]_n && \text{(from } (a)\text{)}\end{aligned}$$

Using (b) and $[T,A] = \partial A$ proves (d). Finally,

$$(\partial A)(z) = \sum_n (\partial A)_n z^{-n-(h_A+1)}. \qquad \text{(from } (d)\text{)}$$

On the other hand,

$$(\partial A)(z) := \frac{d}{dz}A(z) = \frac{d}{dz}\sum_n A_n z^{-n-h_A} = \sum_n -(n+h_A)z^{-n-h_A-1} \iff (\partial A)_n = -(n+h_A)A_n.$$

■

Proof of Proposition 3.2:

$$(-1)^{|A||B|}B(w)A(z) = (-1)^{|A||B|}\sum_n\frac{[B,A]_n(z)}{(w-z)^n} = (-1)^{|A||B|}\sum_n(-1)^n\frac{[B,A]_n(z)}{(z-w)^n}.$$

Hence, we get

$$\sum_n \frac{[A,B]_n(w)}{(z-w)^n} = (-1)^{|A||B|} \sum_n (-1)^n \frac{[B,A]_n(z)}{(z-w)^n}.$$

Integrating both sides with respect to z against $(z-w)^{n-1}$ along a contour C_w enclosing w,

$$\begin{aligned} [A,B]_n(w) &= (-1)^{|A||B|} \oint_{C_w} \frac{dz}{2\pi i}(z-w)^{n-1} \sum_p (-1)^p \frac{[B,A]_p(z)}{(z-w)^p} \\ &= \oint_{C_w} \frac{dz}{2\pi i} \sum_p (-1)^{|A||B|+p} \frac{[B,A]_p(z)}{(z-w)^{p-n+1}}. \end{aligned} \tag{B.15}$$

The sum over p in (B.15) truncates to $p \geq n$ since the integrand becomes analytic for $p < n$. We may relabel the summation index using $l = p - n$. Now, using Cauchy's Integral Formula [30] (one can derive this by Taylor expanding $[B,A]_n(z)$ around w and then using the Residue Theorem)

$$\oint_{C_w} \frac{dz}{2\pi i} \frac{f(z)}{(z-w)^{l+1}} = \frac{1}{l!} \frac{d^l f}{dz^l}\bigg|_{z=w}, \tag{B.16}$$

equation (B.15) now reads

$$[A,B]_n(w) = (-1)^{|A||B|+n}[B,A]_n(w) + (-1)^{|A||B|+n} \sum_{l \geq 1} \frac{(-1)^l}{l!} \partial^l [B,A]_{l+n}(w).$$

Rearranging the above gives the first equality in (3.8). To obtain the second equality, we simply exchange $A \leftrightarrow B$ and multiply by $(-1)^{|A||B|+n}$.

■

Proof of Lemma 3.3:

$$\begin{aligned} 0 &= [\partial \mathbb{1}, A]_{n+1} && \text{(since } \partial\mathbb{1} = 0\text{)} \\ &= -n[\mathbb{1}, A]_n. && \text{(from proposition 3.1}(a)\text{)} \end{aligned}$$

Thus, $[\mathbb{1}, A]_{n \neq 0} = 0$. Now

$$[\mathbb{1}, A]_0 = \oint_{C_w} \frac{dz}{2\pi i} \frac{\mathbb{1}(z)A(w)}{z-w} = \oint_{C_w} \frac{dz}{2\pi i} \frac{\mathrm{id}_{\mathfrak{M}} A(w)}{z-w} = \oint_{C_w} \frac{dz}{2\pi i} \frac{A(w)}{z-w} = A(w)$$

which proves that $[\mathbb{1}, A]_0 = A$.

■

Proof of Proposition 3.4: We will use the integral expressions of the brackets and manipulate contours to prove this proposition. It is imperative to remember that these are

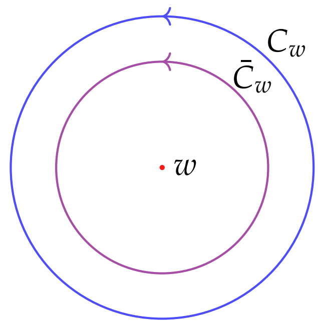

Figure B.1: Illustration of the contours in the integral form of $[A,[B,C]_p]_q(w)$.

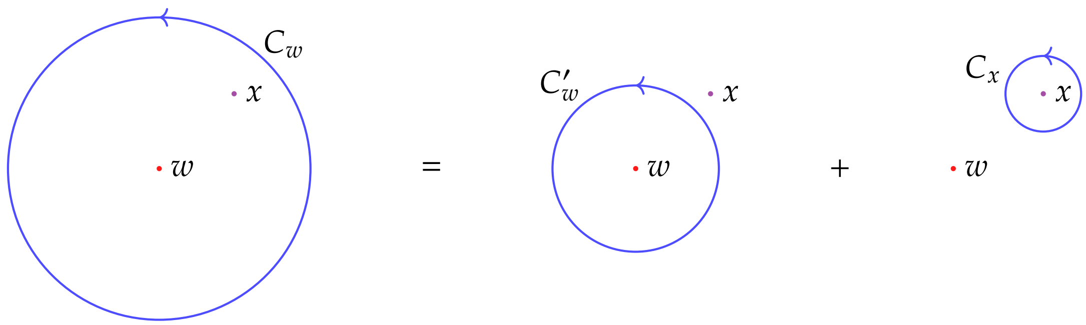

Figure B.2: Visualising the contour manipulations performed on the contours in figure B.1.

expressions containing radially ordered operator products, so when manipulating contours, we must be careful with the ordering of operator products.

$$
\begin{aligned}
[A,[B,C]_p]_q(w) &= \oint_{C_w} \frac{dz}{2\pi i}(z-w)^{q-1}A(z)[B,C]_p(w) \\
&= \oint_{C_w} \frac{dz}{2\pi i} \oint_{\bar{C}_w} \frac{dx}{2\pi i}(z-w)^{q-1}(x-w)^{p-1}A(z)B(x)C(w).
\end{aligned}
$$

Figure B.1 shows the contours C_w and $\bar{C}_w$ centred around w. To make progress, we fix the value of x, perform the z integral (along C_w) and then perform the x integral. We then deform C_w into the sum of two contours as shown in figure B.2.

Note that $|z| < |x|$ when integrating along C'_w contour, so we must reorder the correlator in the integrand accordingly, which yields a Koszul sign $(-1)^{|A||B|}$.

$$
\begin{aligned}
[A,[B,C]_p]_q(w) = &\oint_{\bar{C}_w} \frac{dx}{2\pi i} \oint_{C'_w} \frac{dz}{2\pi i}(z-w)^{q-1}(x-w)^{p-1}(-1)^{|A||B|}B(x)A(z)C(w) \\
&+ \oint_{\bar{C}_w} \frac{dx}{2\pi i} \oint_{C_x} \frac{dz}{2\pi i}(z-w)^{q-1}(x-w)^{p-1}A(z)B(x)C(w)
\end{aligned}
$$

The first line is simple to evaluate using (3.4):

$$\oint_{\bar{C}_w} \frac{dx}{2\pi i} \oint_{C'_w} \frac{dz}{2\pi i}(z-w)^{q-1}(x-w)^{p-1}(-1)^{|A||B|}B(x)A(z)C(w) = (-1)^{|A||B|}[B,[A,C]_q]_p(w). \tag{B.17}$$

We thereby obtain the first term on the RHS of the proposition. Performing the C_x integral requires a bit more effort, since we have a $(z-w)^{q-1}$ term in the integrand. We first expand the $A(z)B(x)$ OPE into a sum over brackets

$$\oint_{\bar{C}_w} \frac{dx}{2\pi i}(x-w)^{p-1}\left(\sum_l \oint_{C_x} \frac{dz}{2\pi i}\frac{(z-w)^{q-1}}{(z-x)^l}[A,B]_l(x)\right)C(w) \tag{B.18}$$

Taylor expanding $(z-w)^{q-1}$ gives

$$(z-w)^{q-1} = \sum_{r\geq 0}\frac{(z-x)^r}{r!}\partial_z^r(z-w)^{q-1}\Big|_{z=x}. \tag{B.19}$$

Since each term in (B.19) contributes a positive power of $z-x$, the sum over l in (B.18) is restricted to $l \geq 1$, as any term with $l \leq 0$ will leave the integrand analytic. For each l, only the $r = l-1$ term from the sum over $r \geq 0$ in the Taylor expansion will contribute to the residue. Consequently, the $1/(l-1)!$ and the factor $(q-1)(q-2)\dots(q-l+1)$ which comes from taking $l-1$ derivatives gives the combinatorial factor $\binom{q-1}{l-1}$ as defined in (3.13). Thus, (B.18) gives

$$\sum_{l\geq 1}\binom{q-1}{l-1}\oint_{\bar{C}_w}\frac{dx}{2\pi i}(x-w)^{p-1+q-l}[A,B]_l(x)C(w) = \sum_{l\geq 1}\binom{q-1}{l-1}[[A,B]_l,C]_{p+q-l}(w). \tag{B.20}$$

Putting together the above and (B.17) gives

$$[A,[B,C]_p]_q(w) = (-1)^{|A||B|}[B,[A,C]_q]_p(w) + \sum_{l\geq 1}\binom{q-1}{l-1}[[A,B]_l,C]_{p+q-l}(w). \tag{B.21}$$

This completes the proof of part (a), given by the equation (3.10). Part (b) can be proven right away by first exchanging $A \leftrightarrow B$, then exchanging $p \leftrightarrow q$ and multiplying by $(-1)^{|A||B|}$. Part (c) requires a separate proof using very similar manipulations of contour integrals so the proof is not shown explicitly to avoid redundancy.

■

Proof Of Proposition 3.5: We have

$$\lim_{w\to 0} A(z)B(w)\mathbb{1} = A(z)B = \sum_n z^{-n-h_A}A_nB.$$

At the same time, the LHS is equal to

$$\lim_{w\to 0}\sum_n \frac{[A,B]_n(w)}{(z-w)^n}\mathbb{1} = \sum_n z^{-n}[A,B]_n.$$

Equating equal powers of z in the two expressions above gives the desired result.

■

Proof Of Proposition 3.6: Using (3.12) we can write $[(AB),C]\in\mathfrak{M}$ as

$$[(AB),C]_q = \sum_{l\geq q}[A,[B,C]_l]_{q-l} + (-1)^{|A||B|}\sum_{l\geq 1}[B,[A,C]_l]_{q-l}. \tag{B.22}$$

Proposition 3.5 and corollary 3.4.2 tell us that

$$\begin{aligned}
[(AB),C] &= (AB)_{q-h_A-h_B}C.\\
[A,[B,C]_l]_{q-l} &= A_{q-l-h_A}B_{l-h_B}C.\\
[B,[A,C]_l]_{q-l} &= B_{q-l-h_b}A_{l-h_A}C.
\end{aligned}$$

Substituting these back into (B.22) gives

$$(AB)_{q-h_A-h_B}C = \left(\sum_{l\geq q}A_{q-l-h_A}B_{l-h_B} + (-1)^{|A||B|}\sum_{l\geq 1}B_{q-l-h_b}A_{l-h_A}\right)C, \quad \forall C\in\mathfrak{M}.$$

We may abstract C since it holds true $\forall C\in\mathfrak{M}$. Relabelling the first summation with $m = q-l-h_A$ and letting $n := q-h_A-h_B$ gives

$$\sum_{l\geq q}A_{q-l-h_A}B_{l-h_B} = \sum_{m\leq -h_A}A_mB_{n-m} = \sum_{l\leq -h_A}A_lB_{n-l}.$$

Relabelling the second summation with $m = l-h_A$ gives

$$\sum_{l\geq 1}B_{q-l-h_b}A_{l-h_A} = \sum_{m\geq -h_A+1}B_{n-m}A_m = \sum_{l>-h_A}B_{n-l}A_l.$$

Putting them together gives us the desired result.

■

Proof Of Proposition 3.7 We act the endomorphism $[A_m,B_n]:\mathfrak{M}\to\mathfrak{M}$ on $C\in\mathfrak{M}$ and use proposition 3.5 twice to get

$$\begin{aligned}
[A_m,B_n]C &:= A_m(B_nC) - (-1)^{|A||B|}B_n(A_mC)\\
&= A_m[B,C]_{n+h_B} - (-1)^{|A||B|}B_n[A,C]_{m+h_A}\\
&= [A,[B,C]_{n+h_B}]_{m+h_A} - (-1)^{|A||B|}[B,[A,C]_{m+h_A}]_{n+h_B}.
\end{aligned}$$

Using (3.10) on the RHS above gives

$$[A_m, B_n]C = \sum_{l\geq 1} \binom{m+h_A-1}{l-1} [[A,B]_l, C]_{m+n+h_A+h_B-l}$$

Corollary 3.4.2 tells us that $[A,B]_l$ has conformal weight $h_A + h_B - l$. We then apply proposition 3.5 on the RHS of the above to get

$$[A_m, B_n]C = \sum_{l\geq 1} \binom{m+h_A-1}{l-1} \left([A,B]_l\right)_{m+n} C. \tag{B.23}$$

Since it holds for any $C \in \mathfrak{M}$, we obtain the desired result.

∎

Proof Of Proposition 3.8 We want to compute the brackets $[T^{bc}, T^{bc}]_{n>0}$.

$$[T^{bc}, T^{bc}]_{n>0} = -\lambda [T^{bc}, (b\partial c)]_{n>0} + (1-\lambda)[T^{bc}, (\partial b c)]_{n>0}. \tag{B.24}$$

We make use of proposition 3.4(a) on each of these terms. This gives the following expressions:

$$[T^{bc}, (b\partial c)]_n = [b, [T^{bc}, \partial c]_n]_0 + \sum_{l\geq 1} \binom{n-1}{l-1} [[T^{bc}, b]_l, \partial c]_{n-l}. \tag{B.25}$$

$$[T^{bc}, (\partial b c)]_1 = 0 + \partial = [\partial b, [T^{bc}, c]_n]_0 + \sum_{l\geq 1} \binom{n-1}{l-1} [[T^{bc}, \partial b]_l, c]_{n-l}. \tag{B.26}$$

It should be evident that the sums truncate by virtue of b and c being primary. Applying proposition 3.1(b) on the first term of (B.25) gives

$$[b, [T^{bc}, \partial c]_n]_0 = (n-1)[b, T^{bc}]$$

We evaluate the above for $n = 1, 2, 4$.

Consider $n = 1$ first. (B.25) and (B.26) become

$$[T^{bc}, (b\partial c)]_1 = 0 + (b\partial c) + (\partial b \partial c) = \partial(b\partial c) \tag{B.27}$$

$$[T^{bc}, (\partial b c)]_1 = 0 + (\partial b \partial c) + (\partial^2 b c) = \partial(\partial b c). \tag{B.28}$$

Thus,

$$[T^{bc}, T^{bc}]_1 = \partial T^{bc}.$$

Now consider $n = 2$. (B.25) and (B.26) now read

$$[T^{bc}, (b\partial c)]_2 = (b\partial c) + (1-\lambda)(b\partial c) + \underbrace{[\partial b, \partial c]_1}_{=0} + \lambda(b\partial c) = 2(b\partial c) \tag{B.29}$$

$$[T^{bc}, (\partial b c)]_2 = (1-\lambda)(\partial b c) + \underbrace{[\partial^2 b, c]_1}_{=0} + (\lambda+1)(b\partial c) = 2(\partial b c). \tag{B.30}$$

Thus,

$$[T^{bc}, T^{bc}]_2 = 2T^{bc}.$$

Finally, for $n = 4$,

$$[T^{bc}, \mathbf{(}b\partial c\mathbf{)}]_4 = [\partial b, \partial c]_3 + 3\lambda[b, \partial c]_2. \tag{B.31}$$

Using proposition 3.1 gives

$$[\partial b, \partial c]_3 = -2[b, \partial c]_2 = -2([b,c]_1 + \partial \underbrace{[b,c]_2}_{=0}) = -2\epsilon\mathbb{1}$$
$$[b, \partial c]_2 = [b,c]_1 + \partial[b,c]_2 = \epsilon\mathbb{1},$$

so we have

$$[T^{bc}, \mathbf{(}b\partial c\mathbf{)}]_4 = \epsilon(3\lambda - 2)\mathbb{1}. \tag{B.32}$$

Performing similar steps for (B.26) with $n = 4$ gives

$$[T^{bc}, \mathbf{(}\partial bc\mathbf{)}]_4 = [\partial^2 b, c]_3 + 3(\lambda+1)[\partial b, c]_2 + 6\lambda[b,c]_1 = \epsilon(3\lambda - 1)\mathbb{1}. \tag{B.33}$$

Putting these together,

$$[T^{bc}, T^{bc}]_4 = \frac{c_{bc}}{2}\mathbb{1} = \epsilon\big(-\lambda(3\lambda-2) + (1-\lambda)(3\lambda-1)\big)\big). \tag{B.34}$$

By the same methods, the other brackets can be shown to vanish. This completes the proof that T^{bc} is a Virasoro element of a general bc-system with central charge $c_{bc} = -2\epsilon(6\lambda^2 - 6\lambda + 1)$.

■

Proof Of Proposition 3.9: For completeness, We first show the derivation of proposition 2.11 using VOAs. We have

$$L_n^{bc} = -2\mathbf{(}b\partial c\mathbf{)}_n - \mathbf{(}\partial bc\mathbf{)}_n. \tag{B.35}$$

We use propositions 3.1(e) and 3.6(b) on the above

$$\begin{aligned}\mathbf{(}b\partial c\mathbf{)}_n &= \sum_{l\le -2} b_l(\partial c)_{n-l} - \sum_{l>-2}(\partial c)_{n-l}b_l = \sum_{l\le 2} -(n-l-1)b_l c_{n-l} + \sum_{l>-2}(n-l-1)c_{n-1}b_l \\ \mathbf{(}\partial bc\mathbf{)}_n &= \sum_{l\le -3}(\partial b)_l c_{n-l} - \sum_{l>-3} c_{n-1}(\partial b)_l = \sum_{l\le -3} -(l+2)b_l c_{n-l} + \sum_{l>-3}(l+2)c_{n-1}b_l.\end{aligned} \tag{B.36}$$

Substituting these back into (B.35) gives

$$
\begin{aligned}
L_n^{bc} &= \sum_{l\le -2} 2(n-l-1)b_l c_{n-l} - 2\sum_{l>-2}(n-l-1)c_{n-1}b_l + \sum_{l\le -3}(l+2)b_l c_{n-l} - \sum_{l>-3}(l+2)c_{n-1}b_l \\
&= \sum_{l\le -3}\big(2(n-l-1)+l+2\big)b_l c_{n-l} + 2(n+1)b_{-2}c_{n+2} - \sum_{l>-2}\big(2(n-l-1)+l+2\big)c_{n-l}b_l \\
&= \sum_{l\le -2}(2n-l)b_l c_{n-1} - \sum_{l>-2}(2n-l)c_{n-l}b_l \\
&= \sum_{l\in\mathbb{Z}}(-2n+l) : c_{n-l}b_l : .
\end{aligned}
\tag{B.37}
$$

Relabelling with $m := l-n$ gives the result

$$L_n^{bc} = \sum_{m\in\mathbb{Z}}(m-n) : c_{-m}b_{m+n} : \tag{B.38}$$

Explicitly, the normal ordered expression is

$$L_n^{bc} = \sum_{m+n>-2}(m-n)c_{-m}b_{m+n} + \sum_{m+n\le -2} b_{m+n}c_{-m}. \tag{B.39}$$

Coming back to the proof at hand, we want to express $d = j_0$ in terms of modes b_n and c_n. We have

$$d = j_0 = (cT^{\mathfrak{M}})_0 + \frac{1}{2}(cT^{bc})_0. \tag{B.40}$$

Proposition 3.6(b) on the first term immediately gives one part of the answer, since

$$(cT^{\mathfrak{M}})_0 = \sum_{l\le 1} c_l L_{-l}^{\mathfrak{M}} + \sum_{l>1} L_{-l}^{\mathfrak{M}} c_l = \sum_{l\in\mathbb{Z}} c_{-l}L_l^{\mathfrak{M}}. \tag{B.41}$$

The last equality follows trivially since $c_n \in \operatorname{End}\Lambda_\infty^{\bullet}$ while $L_n \in \operatorname{End}\mathfrak{M}$, so these endomorphisms commute. However, we have to work harder to express the second term in j_0 in terms of b_n and c_n.

$$
\begin{aligned}
(cT^{bc})_0 &= \sum_{l\le 1} c_l L_{-l}^{bc} + \sum_{l>1} L_{-l}^{bc} c_l \\
&= \sum_{l\le 1} c_l \left(\sum_{k-l>-2}(k+l)c_{-k}b_{k-l}\right) - \sum_{l\le 1}\left(\sum_{k-l\le -2}(k+l)b_{k-l}c_{-k}\right)c_l \\
&\quad + \sum_{l>1}\left(\sum_{k-l>-2}(k+l)c_{-k}b_{k-l}\right)c_l - \sum_{l\le 1}\left(\sum_{k-l\le -2}(k+l)b_{k-l}c_{-k}\right)c_l.
\end{aligned}
\tag{B.42}
$$

In the first term and fourth terms, c_l can be commuted past b_{k-l} and contribute only a negative sign since any non-zero anti-commutator between c_l and b_{k-l} would require $k = 0$.

This is not possible in either of the sums because they force k to be at most -1 and at least 1 respectively. Hence, we have

$$\begin{aligned}(cT^{bc})_0 &= \sum_{l\in\mathbb{Z}}\sum_{k-l>-2}(k+l)c_l c_{-k} b_{k-l} - \sum_{l\in\mathbb{Z}}\sum_{k-l\leq-2}(k+l)b_{k-l}c_{-k}c_l \\ &= \sum_{l\in\mathbb{Z}}\sum_{k+l>-2}(k-l)c_{-l}c_{-k}b_{k+l} + \sum_{l\in\mathbb{Z}}\sum_{k+l\leq-2}(k-l)b_{k-l}c_{-l}c_{-k},\end{aligned} \tag{B.43}$$

where to obtain the last equality, we have relabelled $l \to -l$ and commuted c_{-l} and c_{-k} in the second term. Finally, we notice the antisymmetry of the c_n modes implies we have a "double counting" in the sum over k. To illustrate this, consider the first term on the RHS of (B.43) and split the sum over k into one over $k > l$ and one over $k < l$ (the $k = l$ terms is zero):

$$\begin{aligned}\sum_{l\in\mathbb{Z}}\sum_{k+l>-2}(k-l)c_{-l}c_{-k}b_{k+l} &= \sum_{l\in\mathbb{Z}}\left(\sum_{\substack{k+l>-2\\k>l}}-(l-k)c_{-l}c_{-k}b_{k+l} + \sum_{\substack{k+l>-2\\k<l}}(k-l)c_{-l}c_{-k}b_{k+l}\right) \\ &= \sum_{l\in\mathbb{Z}}\left(\sum_{\substack{k+l>-2\\l>k}}-(k-l)c_{-k}c_{-l}b_{k+l} + \sum_{\substack{k+l>-2\\k<l}}(k-l)c_{-l}c_{-k}b_{k+l}\right) \\ &= \sum_{l\in\mathbb{Z}}\left(\sum_{\substack{k+l>-2\\l>k}}(k-l)c_{-l}c_{-k}b_{k+l} + \sum_{\substack{k+l>-2\\k<l}}(k-l)c_{-l}c_{-k}b_{k+l}\right) \\ &= 2\sum_{l\in\mathbb{Z}}\sum_{\substack{k+l>-2\\l>k}}(k-l)c_{-l}c_{-k}b_{k+l}.\end{aligned} \tag{B.44}$$

The second equality is obtained by swapping labels $k \leftrightarrow l$ and the third equality by commuting c_{-k} and c_{-l}, both done in the first term. The same can be done for the other term in the RHS of (B.43), so we have

$$\begin{aligned}\frac{1}{2}(cT^{bc})_0 &= \sum_{l\in\mathbb{Z}}\left(\sum_{\substack{k+l>-2\\l>k}}(k-l)c_{-l}c_{-k}b_{k+l} + \sum_{\substack{k+l\leq-2\\k<l}}(k-l)b_{k-l}c_{-l}c_{-k}\right) \\ &= \sum_{\substack{l,k\in\mathbb{Z}\\k<l}}(k-l) : b_{k-l}c_{-l}c_{-k} : \ .\end{aligned} \tag{B.45}$$

Putting everything together (changing letters to $m = l$, $n = k$ above), we obtain

$$d = \sum_n c_{-n}L_n + \sum_{\substack{n,m\in\mathbb{Z}\\n<m}}(n-m) : b_{m+n}c_{-m}c_{-n} : \ . \tag{B.46}$$

■

Proof Of Proposition 3.10: Keep in mind that here, T^{bc} is the Virasoro element of the VOA of a fermionic bc-system with $\lambda = 2$. By definition of d,

$$db = [j, b]_1 = [(cT)^{\mathfrak{M}}, b]_1 + \frac{1}{2}[(cT^{bc},)b]_1. \tag{B.47}$$

Applying (3.10) to the RHS gives

$$\begin{aligned} db &= \sum_{l\geq 1}[c, [T^{\mathfrak{M}}, b]_l]_{1-l} + \sum_{l} \geq 1[T^{\mathfrak{M}}, [c, b]_l]_{1-l} \\ &= \frac{1}{2}\sum_{l\geq 1}[c, [T^{bc}, b]_l]_{1-l} + \frac{1}{2}\sum_{l\geq 1}[T^{bc}, [c, b]_l]_{1-l} \\ &= 0 + T + \frac{1}{2}(c\partial b) + [c, b]_{-1} + \frac{1}{2}T^{bc}. \end{aligned}$$

Proposition 3.1(a) tells us that $[c, b]_{-1} = (\partial cb)$ and proposition 3.2 tells us that

$$(c\partial b) = -(\partial bc) \qquad (\partial cb) = -(b\partial c).$$

This leaves us with

$$db = T^{\mathfrak{M}} - \frac{1}{2}(\partial bc) - (b\partial c) + \frac{1}{2}T^{bc} = T + T^{bc}. \tag{B.48}$$

Now

$$L_0^{\text{tot}} = (db)_0 = \big([j, b]_1\big)_0 = \sum_{l\geq 1}\binom{0}{l-1}\big([j, b]_1\big)_0 \tag{B.49}$$

By proposition 3.7, the RHS is equal to $j_0 b_0 - (-1)^{|j|} b_0 j_0 = [d, b_0]$, which completes the proof.

■

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