1. Gauge theory & principal bundles

1.1 Maxwell's equations:

$$dF = 0 \quad , \quad d * F = J$$

$$\Omega^2(\mathbb{R}^4) \ni F \longleftrightarrow (\vec{E}, \vec{B}) \quad \text{EM fields}$$

$$dF = 0 \Rightarrow F = dA \quad \text{on } \mathbb{R}^4$$

field strength

gauge potential

$$A \longmapsto A + d\lambda$$

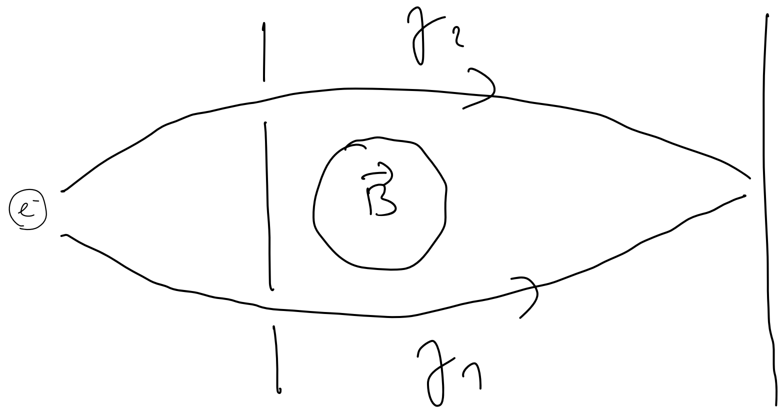
$\in \Omega^1(\mathbb{R}^4)$

gauge sym. (redundancy)

Q: Is A just a mathematical gadget or does it have phys. significance?

1.2 Aharonov - Bohm effect :

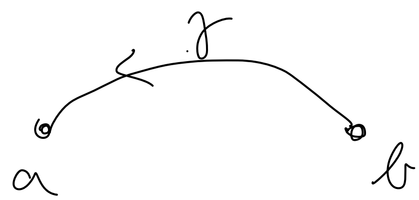
- double-slit experiment w/ a solenoid



electron wavefunctions acquire a phase $e^{i\varphi_{1,2}}$

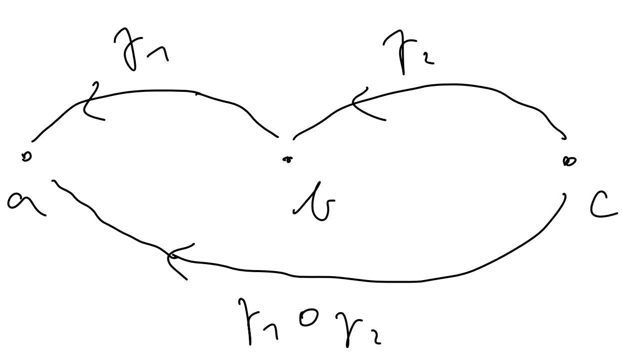
$$\varphi_{1,2} = \int_{\gamma_{1,2}} A$$

- lesson : Nature associates to each path γ a phase $e^{i\int_\gamma A}$



$\longrightarrow \quad e^{i\int_\gamma A}$

- phases add when paths are concatenated



$\longrightarrow \quad e^{i\int_{\gamma_1} A + i\int_{\gamma_2} A}$

- phases invert when paths are inv.
- constant path mapped to 1.

$\Rightarrow$ association functorial

- only paths w/ matching endpoints can be composed $\Rightarrow$ space of paths

$\to$ path groupoid $P\mathbb{R}^4$

$\to$ $BU(1)$

- generalise : $\mathbb{R}^4 \longrightarrow M$ (manifold) smooth, contractible

$U(1) \longrightarrow G$ (Lie group)

<u>gauge theory</u> $\approx$ $\underbrace{\text{functor } PM \longrightarrow BG}_{\text{parallel transport of point-like particles along one-dim. paths}}$

1.3 Category theory (elements):

Def: a category $\mathcal{C} = (C_1 \overset{s}{\underset{t}{\rightrightarrows}} C_0)$ consists of:

- coll. of objects C_0
- coll. of morphisms C_1
- source s, target t, identity id maps
- associative & unital composition map $\circ$

$$\begin{tikzcd}
a \arrow[loop left, "id_a"] \arrow[r, "f"] \arrow[dr, "g \circ f"'] & b \arrow[loop right, "id_b"] \arrow[d, "g"] \\
 & c \arrow[loop right, "id_c"]
\end{tikzcd}$$

Groupoid = category w/ <u>invertible</u> morphisms

$\left(\forall f \in C_1 \ \exists f^{-1} \in C_1 : f \circ f^{-1} = id_{t(f)} \ \& \ f^{-1} \circ f = id_{s(f)} \right)$

Examples : • $\mathcal{P}M$ — path grpd

• group $G \longleftrightarrow$ delooping BG

$$BG = (G \rightrightarrows *)$$

$\longrightarrow$ single - object grpd

• Čech grpd $\check{C}(\{U_a\})$

cover $\{U_a\}$ of manifold M

$$\check{C}(\{U_a\}) = \Big(\coprod_{a,b} U_{ab} \rightrightarrows \coprod_{a} U_a \Big)$$

patches — double overlaps — disjoint union

$$\begin{tikzcd}
(x,a) & & (x,b) \arrow[ll, "{(x,a,b)}"'] \\
& & (x,c) \arrow[ull, "{(x,a,c)}"] \arrow[u, "{(x,b,c)}"']
\end{tikzcd}$$

Def: Functor = morphism between categories

$$\begin{tikzcd}
a \arrow[loop left, "id_a"] \arrow[r, "f"] \arrow[dr, "g \circ f"'] & b \arrow[loop right, "id_b"] \arrow[d, "g"] & & F(a) \arrow[loop left, "id_{F(a)}"] \arrow[r, "F(f)"] \arrow[dr, "F(g)\circ F(f)"'] & F(b) \arrow[loop right, "id_{F(b)}"] \arrow[d, "F(g)"] \\
& c \arrow[loop right, "id_c"] & \arrow[r, "F"] & & F(c) \arrow[loop right, "id_{F(c)}"]
\end{tikzcd}$$

Def. Natural transformation

= morphism between functors

$$\begin{tikzcd} \mathcal{C} \arrow[r, bend left=60, "F"] \arrow[r, bend right=60, "G"'] \arrow[r, phantom, "\Downarrow \alpha"] & \mathcal{D} \end{tikzcd} \quad : \quad C_0 \ni x \longmapsto \alpha_x \in D_1$$

$$\begin{tikzcd} F(x) \arrow[r, "F(f)"] \arrow[d, "\alpha_x"'] & F(y) \arrow[d, "\alpha_y"] \\ G(x) \arrow[r, "G(f)"] & G(y) \end{tikzcd}$$

Examples:

- holonomy / parallel transport functor $PM \longrightarrow BG$
- principal G-bundle over M : $\check{\mathcal{C}}(\{U_a\}) \longrightarrow BG$

1.4 Principal G-bundles (1-bundles)

$P \to M$, $R : P \times G \to P$, G-equiv. loc. triv.

M - manifold, cover with patches $\{U_a\}$

G-bun. over M encoded in functor

$$\check{\mathcal{E}}(\{U_a\}) \longrightarrow BG$$

$$\begin{tikzcd} \bigsqcup_{a,b} U_{ab} \arrow[r, "g"] \arrow[d, shift left] \arrow[d, shift right] & G \arrow[d, shift left] \arrow[d, shift right] \\ \bigsqcup_a U_a \arrow[r, "e"] & * \end{tikzcd} \longleftrightarrow$$

transition functions g_{ab}

$$\underbrace{g_{ab}\, g_{bc} = g_{ac}}_{\text{cocycle condition}}$$

- gauge trafos = natural isomorphisms

coboundaries $\bigsqcup_a U_a \xrightarrow{\gamma} G$

$$g_{ab}\, \gamma_b = \gamma_a\, \tilde{g}_{ab}$$

$\to$ equivalence of G-bundles

- Connection on a G-bun.

3 equivalent definitions:

1.) G-equiv. horizontal distribution on $P \to M$

$$T_p P = H_p P \oplus V_p P \,, \quad (R_g)_* H_p P = H_{pg} P \,, \quad p \in P, \, g \in G$$

2.) $\mu \in \Omega^1(P) \otimes \mathfrak{g}$:

a) $R_g^* \mu = Ad_{g^{-1}} \mu \quad , \, g \in G$

b) $\mu(X^R) = X \quad , \, X \in \mathfrak{g} = Lie(G)$

(fund. vec. field of R: $X_p^R := \frac{d}{dt}\Big|_0 p \exp(tX)$, $\exp: \mathfrak{g} \to G$)

3.) a collection of $\mathfrak{g}$-valued 1-forms on patches $\{ A_i \in \Omega^1(U_i) \otimes \mathfrak{g} \}$ s.t. on $U_{ij} \neq \emptyset$

$$A_j = g_{ij}^{-1} A_i g_{ij} + \underbrace{g_{ij}^{-1} d g_{ij}}_{\text{(left) Maurer-Cartan 1-form}}$$

gauge trafos $\gamma : \bigsqcup_i U_i \to G$

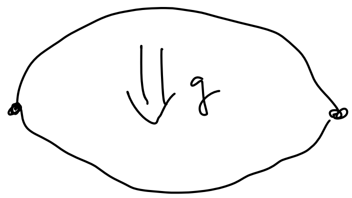
$$\tilde{A}_i = \gamma_i^{-1} A_i \gamma_i + \gamma_i^{-1} d \gamma_i$$

Deligne cocycles (g, A)

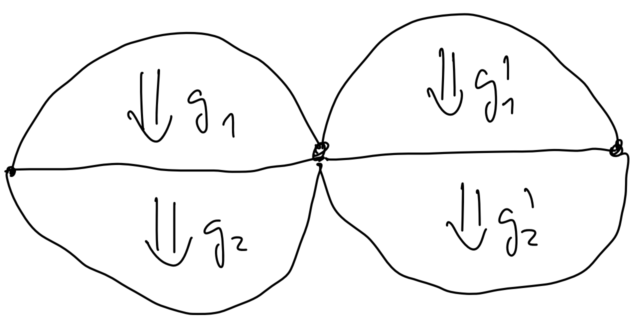
2. Higher gauge theory & higher PB's

2.1 Higher parallel transport

- consider parallel transport of strings w/ fixed endpoints



- consistency dictates



$$\Leftrightarrow \quad (g_2 g_1)(g_2' g_1') = (g_2 g_2')(g_1 g_1')$$

- $g_2 = 1 = g_1' \implies g_1 g_2' = g_2' g_1$

  $\implies$ G abelian (Eckmann–Hilton)

  $\to$ PROBLEM!

- SOLUTION: higher categories!

2.2 Strict 2-categories

- categorify categories!
- introduce 2-morphisms & horizontal composition $\otimes$

functor

$$
\begin{tikzcd}
a & b \arrow[l, bend right=50, "f_1"'] \arrow[l, bend left=50, "f_2"] \arrow[l, phantom, "\Downarrow\alpha"] & c \arrow[l, bend right=50, "g_1"'] \arrow[l, bend left=50, "g_2"] \arrow[l, phantom, "\Downarrow\beta"] & {} \arrow[r, "\otimes"] & {} & a & c \arrow[l, bend right=50, "f_1\otimes g_1"'] \arrow[l, bend left=50, "f_2\otimes g_2"] \arrow[l, phantom, "\Downarrow\alpha\otimes\beta"]
\end{tikzcd}
$$

- such that

$$
\begin{tikzcd}[row sep=large]
a & b \arrow[l, bend right=60, "f_1"'] \arrow[l, "f_2" description] \arrow[l, bend left=60, "f_3"] \arrow[l, phantom, shift right=3ex, "\Downarrow\alpha_1"] \arrow[l, phantom, shift left=3ex, "\Downarrow\alpha_2"] & c \arrow[l, bend right=60, "g_1"'] \arrow[l, "g_2" description] \arrow[l, bend left=60, "g_3"] \arrow[l, phantom, shift right=3ex, "\Downarrow\beta_1"] \arrow[l, phantom, shift left=3ex, "\Downarrow\beta_2"]
\end{tikzcd}
$$

vertical comp.

$$\Longleftrightarrow \quad (\alpha_2\circ\alpha_1)\otimes(\beta_2\circ\beta_1) = (\alpha_2\otimes\beta_2)\circ(\alpha_1\otimes\beta_1)$$

interchange law $\longrightarrow$ relaxes Eckmann – Hilton

- $\mathcal{C} = (C_2 \rightrightarrows C_1 \rightrightarrows C_0)$

2-morphisms 1-category objects

1-morphisms

$\mathcal{C}(a,b)$ – category of 1- and 2-morphisms from $b \in C_0$ to $a \in C_0$

- definition admits weaker versions

- a 2-functor (normalised pseudofunc.) Φ between 2-categories $\mathcal{C}$ & $\tilde{\mathcal{C}}$ is a triple of maps (ϕ_0, ϕ_1, ϕ_2):

$$\begin{tikzcd} a & b \arrow[l, bend right=50, "f_1"'] \arrow[l, bend left=50, "f_2"] \arrow[l, phantom, "\Downarrow\alpha"] & c \arrow[l, bend right=50, "g_1"'] \arrow[l, bend left=50, "g_2"] \arrow[l, phantom, "\Downarrow\beta"] \end{tikzcd} \xrightarrow{\ \Phi\ } \begin{tikzcd} \phi_0(a) & \phi_0(b) \arrow[l, bend right=50, "\phi_1^{ab}(f_1)"'] \arrow[l, bend left=50, "\phi_1^{ab}(f_2)"] \arrow[l, phantom, "\Downarrow\phi_1^{ab}(\alpha)"] & \phi_0(c) \arrow[l, bend right=50, "\phi_1^{bc}(g_1)"'] \arrow[l, bend left=50, "\phi_1^{bc}(g_2)"] \arrow[l, phantom, "\Downarrow\phi_1^{bc}(\beta)"] \end{tikzcd}$$

& $\phi_1^{ab}(\alpha) \tilde{\otimes} \phi_1^{bc}(\beta) \xRightarrow{\ \phi_2^{abc}\ } \phi_1^{ac}(\alpha \otimes \beta)$

& . . .

- Similarly, can define
  natural 2-transformations $\longleftrightarrow$ gauge transf.
  &
  2-modifications $\longleftrightarrow$ higher gauge transf.

$$
\begin{tikzcd}[column sep=huge]
\mathcal{C} \arrow[r, bend left=60, "\Phi"{name=U}] \arrow[r, bend right=60, "\Psi"'{name=D}] & \tilde{\mathcal{C}}
\end{tikzcd}
\qquad \alpha \overset{\varphi}{\Rrightarrow} \beta
$$

Examples:

- **2-groupoid**: all morphisms invertible
  (2-mor. inv. w.r.t. both $\circ$ & $\otimes$)

- **Čech 2-groupoid**: $\check{\mathcal{C}}(\{U_a\}) = \left(\bigsqcup_{a,b} U_{ab} \rightrightarrows \bigsqcup_{a,b} U_{ab} \rightrightarrows \bigsqcup_a U_a \right)$

- (strict) **2-group** = (strict) 2-groupoid w/ a single object

$$\left(G_2 \rightrightarrows G_1 \rightrightarrows * \right)$$

2.3 Crossed modules of groups

Def. A **crossed mod. of grps** is given by:

- a pair of groups (G, H)
- left action $\triangleright : G \to \mathrm{Aut}(H)$
- homomorphism $t : H \to G$

s.t. $t(g \triangleright h) = g\, t(h)\, g^{-1}$ & $\underbrace{t(h_1) \triangleright h_2 = h_1 h_2 h_1^{-1}}_{\text{Peiffer id.}}$

NOT.: $\mathcal{G} := (H \xrightarrow{t} G, \triangleright)$

Thm. A strict 2-group is equivalent to a crossed module of groups.

Proof: Given $(H \xrightarrow{t} G, \triangleright)$ define 2-group:

- $(G \ltimes H \rightrightarrows G)$ $\quad | \quad g \xleftarrow{(g,h)} t(h^{-1})g$

- $g_1 \otimes g_2 := g_1 g_2$, $(g_1, h_1) \otimes (g_2, h_2) := (g_1 g_2, (g_1 \triangleright h_2) h_1)$

  $(g_1, h_1) \circ (t(h_1^{-1}) g_1, h_2) := (g_1, h_1 h_2)$

Conversely, given $(G_2 \overset{t}{\underset{s}{\rightrightarrows}} G_1)$, define crossed mod.:

- $G := G_1$ & $H := \ker(s)$
- $g_1 g_2 := g_1 \otimes g_2$, $h_1 h_2 := h_1 \circ (h_2 \otimes \mathrm{id}_{s(h_1)}) = h_2 \otimes h_1$,

$t(h) := s(h^{-1})$, $g \triangleright h := \mathrm{id}_g \otimes h \otimes \mathrm{id}_{g^{-1}}$

$\uparrow$

$h^{-1} \otimes h = \mathrm{id}$

$\square$

Morphisms of crossed modules = butterflies

$$
\begin{tikzcd}
H_1 \arrow[dd, "t_1"'] \arrow[dr, "\kappa_1"] & & H_2 \arrow[dl, "\kappa_2"'] \arrow[dd, "t_2"] \\
& E \arrow[dl, "\gamma_1"'] \arrow[dr, "\gamma_2"] & \\
G_1 & & G_2
\end{tikzcd}
$$

blue diag. is a short exact seq.

$+ \;\ldots$

- both diags short exact seq.

$\Rightarrow$ $\underbrace{\text{flippable butterfly}}_{\text{equivalence of crossed modules}}$

Examples:

- $(U(1) \to 1) \cong BU(1)$ – underlies abelian gerbes
- $G \cong (1 \hookrightarrow G, \mathrm{id})$ – underlies G-bundles
- $\mathcal{L}G := (L_0 G \hookrightarrow P_0 G, \mathrm{Ad})$

$* \; P_0 G := \{ p \in \mathcal{C}^\infty([0,1], G) \mid p(0) = 1 \}$

* $b : P_0G \longrightarrow G \qquad \rho \mapsto \rho(1)$

* $L_0G = \ker(b)$

In fact, $\mathcal{L}G \cong G$ because

$$\begin{tikzcd}
1 \arrow[dd] \arrow[dr] & & L_0G \arrow[dl] \arrow[dd] \\
& P_0G \arrow[dl, "b"'] \arrow[dr, "id"] & \\
G & & P_0G
\end{tikzcd}$$

is a flip. butterfly!

2.4 Principal 2-bundles

M - manifold, cover w/ patches $\{U_a\}$

$\mathcal{G} = (H \xrightarrow{t} G, \triangleright)$ - crossed module

$\Rightarrow B\mathcal{G} := (G \ltimes H \rightrightarrows G \rightrightarrows *)$

Def. A **principal $\mathcal{G}$-bundle** = principal 2-bundle w/ structure 2-group given by $\mathcal{G}$ is a functor

$$\check{\mathcal{C}}(\{U_a\}) \longrightarrow B\mathcal{G}$$

$$\begin{pmatrix} \bigsqcup_{a,b} U_{ab} \\ \downarrow\downarrow \\ \bigsqcup_{a,b} U_{ab} \\ \downarrow\downarrow \\ \bigsqcup_{a} U_{a} \end{pmatrix} \xrightarrow{\quad \Phi \quad} \begin{pmatrix} G \ltimes H \\ \downarrow\downarrow \\ G \\ \downarrow\downarrow \\ * \end{pmatrix}$$

Def. of 2-functor boils down to the maps:

- $e_a := \phi_0|_{U_a} : U_a \to *$
- $g_{ab} := \phi_1|_{U_{ab}} : U_{ab} \to G$
- $m_{abc} \equiv (g_{abc}, h_{abc})$
  $:= \phi_2|_{U_{abc}} : U_{abc} \to G \ltimes H$

$$g_{ab} \otimes g_{bc} \overset{m_{abc}}{\Longrightarrow} g_{ac}$$

+ consistency cond.

$$m_{acd} \circ (m_{abc} \otimes id_{g_{cd}}) = m_{abd} \circ (id_{g_{ab}} \otimes m_{bcd})$$

$\Rightarrow$ $t(m_{abc}) = g_{abc} = g_{ac}$ &

$s(m_{abc}) = t(h^{-1}_{abc})\, g_{abc} = g_{ab} \otimes g_{bc} = g_{ab} g_{bc}$ &

$$(g_{ad}, h_{acd}) \circ (g_{ac} g_{cd}, h_{abc}) = (g_{ad}, h_{abd}) \circ (g_{ab} g_{bd}, g_{ab} \rhd h_{bcd})$$

$\Rightarrow$

Thm. A principal $\mathcal{G}$-bundle can be equivalently described relative to covering $\{U_a\}$ via (degree 2) Čech cocycles $(\{g_{ab}\}, \{h_{abc}\})$, i.e.

$$g_{ab} : U_{ab} \longrightarrow G \quad , \quad h_{abc} : U_{abc} \longrightarrow H$$

s.t.

$$t(h_{abc})\, g_{ab} g_{bc} = g_{ac} \;, \quad h_{acd} h_{abc} = h_{abd} (g_{ab} \rhd h_{bcd})$$

Similarly, natural 2-isomorphisms produce gauge transformations, desc. by coboundaries $g_a : U_a \to G$ and $h_{ab} : U_{ab} \to H$ s.t.

$$g_a \tilde{g}_{ab} = t(h_{ab})\, g_{ab}\, g_b \qquad \&$$

$$h_{ac}\, h_{abc} = (g_a \triangleright \tilde{h}_{abc})\, h_{ab}\, (g_{ab} \triangleright h_{bc}) .$$

Coboundaries can be in turn related by higher coboundaries obtained from 2-modifications : $h_a : U_a \to H$

$$\tilde{g}_a = t(h_a)\, g_a \quad \& \quad \hat{h}_{ab} = h_a h_{ab} (g_{ab} \triangleright h_b^{-1})$$

N.B. • Principal $\mathcal{G}$-bundles = non-abelian gerbes

• for $\mathcal{G} = BU(1) = (U(1) \to 1)$ we recover abelian gerbe characterised by $H^3(M, \mathbb{Z})$

3. Connections on non-abelian gerbes

3.1 L_∞-algebras

<u>Def.</u> An L_∞-algebra is:

- a $\mathbb{Z}$-graded vector space
  $$L = \bigoplus_{n \in \mathbb{Z}} L_n$$
- graded-antisymmetric multilin. maps (higher products)
  $$\mu_i : \Lambda^i L \longrightarrow L$$
  of degree $|\mu_i| = 2 - i$, $i \in \mathbb{N}$
- which satisfy homotopy Jacobi identities

e.g.:
$$0 = \mu_1 \circ \mu_1$$
$$0 = \mu_1(\mu_2(\ell_1, \ell_2)) - \mu_2(\mu_1(\ell_1), \ell_2) + (-1)^{|\ell_1||\ell_2|} \mu_2(\mu_1(\ell_2), \ell_1)$$
$$\vdots$$

- $L = L_{-(n-1)} \oplus \dots \oplus L_0 \to$ Lie n-algebra

- $\underline{\mu_1 = 0}$: skeletal / minimal

  $\underline{\mu_i = 0 \,,\, i \geq 3}$: strict

- can formulate the notion of morphisms of L_∞-algebras

- equivalences = quasi-isomorphisms
  (isomorphism on μ_1-cohomology)

<u>Thm.</u> Any L_∞-alg. is quasi-isomorphic to a skeletal one and a strict one.

<u>Examples:</u>

- crossed modules of Lie algebras
  $\leftarrow$ differentiate crossed modules of Lie groups

$(\underline{h} \xrightarrow{t} \mathfrak{g}, \triangleright)$: $t(X \triangleright Y) = [X, t(Y)]$, $t(Y_1) \triangleright Y_2 = [Y_1, Y_2]$

view as a <u>strict Lie 2-algebra</u>:

$$L = \underset{\substack{\uparrow \\ L_{-1}}}{\underline{h}} \oplus \underset{\substack{\uparrow \\ L_0}}{\mathfrak{g}} \,, \quad \mu_1 = t \,, \quad \mu_2 = \begin{cases} [\cdot,\cdot] \,, & \mathfrak{g} \times \mathfrak{g} \\ \cdot \triangleright \cdot \,, & \mathfrak{g} \times \underline{h} \end{cases}$$

- string Lie 2-algebra $(\mathfrak{g}, [\cdot,\cdot], \langle\cdot,\cdot\rangle)$

  * <u>skeletal model</u>:

$$\mathrm{string}^0(\mathfrak{g}) = (\mathbb{R} \xrightarrow{0} \mathfrak{g})$$

$$\mu_2 = [\cdot,\cdot] \ , \ \mathfrak{g}\times\mathfrak{g}$$

$$\mu_3 = \langle\cdot,[\cdot,\cdot]\rangle \ , \ \mathfrak{g}\times\mathfrak{g}\times\mathfrak{g}$$

  * <u>strict model</u>:

$$\mathrm{string}^1(\mathfrak{g}) = \left(L_0\mathfrak{g}\oplus\mathbb{R} \xrightarrow{\mathrm{proj}_1} P_0\mathfrak{g} \ , \ \rhd\right)$$

$(L_0\mathfrak{g} \hookrightarrow P_0\mathfrak{g}, \mathrm{ad})$ – crossed mod. of Lie algebras from $\mathcal{L}G$

$$\alpha \rhd (\gamma, q) = \left([\alpha,\gamma] \ , \ \underbrace{\frac{1}{2\pi}\int_0^1 d\tau \left\langle \frac{\partial\alpha}{\partial\tau}, \gamma\right\rangle}_{\text{Kac-Moody 2-cocycle}}\right)$$

<u>3.2</u> Higher gauge theory from L_∞-alg.

- L_∞-alg. come with their own higher gauge theory called homotopy

Maurer – Cartan theory

- (L, μ_i) – L_∞-alg.
  - $a \in L_1$ – gauge potential
  - $f(a) := \mu_1(a) + \frac{1}{2}\mu_2(a,a) + \frac{1}{3!}\mu_3(a,a,a) + \dots$

    $\in L_2$

    → field strength or curvature
  - Bianchi identity:

$$\sum_{i \geq 0} \frac{1}{i!}\mu_{i+1}(a, \dots, a, f(a)) = 0.$$

  - Maurer – Cartan elements: $f(a) = 0$
  - gauge trafos: $c_0 \in L_0$

$$\delta_{c_0} a = \sum_{i \geq 0} \frac{1}{i!}\mu_{i+1}(a, \dots, a, c_0)$$

  - higher gauge trafos: $c_{-1} \in L_{-1}$

$$\delta_{c_{-1}} c_0 = \sum_{i \geq 0} \frac{1}{i!}\mu_{i+1}(a, \dots, a, c_{-1})$$

$$\vdots$$

* Cyclic structure $\langle\cdot,\cdot\rangle : L\times L \to \mathbb{R}$

deg. -3

* homotopy Maurer-Cartan action

$$S_{hMC} = \sum_{i=1}^{\infty} \frac{1}{(i+1)!} \langle a, \mu_i(a,\dots,a)\rangle$$

3.3 Local connections

M - contractible manifold (e.g. a patch)

$(\Omega^\bullet(M), d)$ - de Rham complex

(L, μ_i) - L_∞-alg.

$\to$ form $\Omega^\bullet(M,L) := \bigoplus_{k\in\mathbb{Z}} \underbrace{(\Omega^\bullet(M)\otimes L)_k}_{(\Omega^0(M)\otimes L_k)\oplus(\Omega^1\otimes L_{k-1})\oplus\dots}$

$\to$ $\Omega^\bullet(M,L)$ is an L_∞-alg.!

$$\hat\mu_1(\alpha_1\otimes l_1) := d\alpha_1\otimes l_1 + (-1)^{|\alpha_1|}\alpha_1\otimes\mu_1(l_1)$$

$$\hat\mu_i(\alpha_1\otimes l_1,\dots,\alpha_i\otimes l_i) := \pm(\alpha_1\wedge\dots\wedge\alpha_i)\otimes\mu_i(l_1,\dots,l_i)$$

For <u>Lie 2-algebra</u> $L = L_{-1} \oplus L_0$:

- $\Omega^1(M, L) \ni a = \underset{\underset{\Omega^1(M)\otimes L_0}{\cap}}{A} + \underset{\underset{\Omega^2(M)\otimes L_{-1}}{\cap}}{B}$

- $F := dA + \frac{1}{2}\mu_2(A, A) + \mu_1(B) \in \Omega^2(M)\otimes L_0$

  $H := dB + \mu_2(A, B) - \frac{1}{3!}\mu_3(A, A, A)$

  $\in \Omega^3(M)\otimes L_{-1}$

- Bianchi : $0 = dF + \mu_2(A, F) - \mu_1(H)$

  $0 = dH + \mu_2(A, H) - \mu_2(F, B)$

  $- \frac{1}{2}\mu_3(A, A, F)$

- gauge trafos : $\alpha \in \Omega^0(M)\otimes L_0$

  $\lambda \in \Omega^1(M)\otimes L_{-1}$

$$\delta A = d\alpha + \mu_2(A, \alpha) - \mu_1(\lambda)$$

$$\delta B = d\lambda + \mu_2(A, \lambda) + \mu_2(B, \alpha) + \frac{1}{2}\mu_3(A, A, \alpha)$$

- higher gauge trafos : $\theta \in \Omega^0(M) \otimes L_{-1}$

$$\delta \alpha = \mu_1(\theta) \quad , \quad \delta \lambda = d\theta + \mu_2(A, \theta)$$

3.4 Principal $\mathcal{G}$ - bundles with connection

M - manifold , cover w/ patches $\{U_i\}$

$\mathcal{G} = (H \xrightarrow{t} G , \triangleright)$ - crossed module of Lie groups

$\mathrm{Lie}(\mathcal{G}) = (\underline{h} \xrightarrow{t} \mathfrak{g} , \triangleright)$ - crossed mod. of Lie algebras

<u>cocycle description</u> :

- cocycles :

$$h_{ijk} \in C^\infty(U_{ijk}, H)$$

$$(g_{ij}, \Lambda_{ij}) \in C^\infty(U_{ij}, G) \oplus \Omega^1(U_{ij}, \underline{h})$$

$$(A_i, B_i) \in \Omega^1(U_i, \mathfrak{g}) \oplus \Omega^2(U_i, \underline{h})$$

such that

$$h_{ikl}\, h_{ijk} = h_{ijl}\,(g_{ij} \triangleright h_{jkl})$$

$$g_{ik} = t(h_{ijk})\, g_{ij}\, g_{jk}$$

$$\Lambda_{ik} = \Lambda_{jk} + g_{jk}^{-1} \triangleright \Lambda_{ij} - g_{ik}^{-1} \triangleright \left(h_{ijk} \nabla_i h_{ijk}^{-1} \right)$$

$$A_j = g_{ij}^{-1} A_i g_{ij} + g_{ij}^{-1} d g_{ij} - t(\Lambda_{ij})$$

$$B_j = g_{ij}^{-1} \triangleright B_i + d\Lambda_{ij} + A_j \triangleright \Lambda_{ij} + \frac{1}{2} [\Lambda_{ij}, \Lambda_{ij}]$$

where $\nabla_i = d + A_i \triangleright$.

- curvatures:

$$F_i = dA_i + \frac{1}{2}[A_i, A_i] + t(B_i)$$

$$H_i = dB_i + A_i \triangleright B_i$$

fake curvature

PROBLEM: COCYCLE DESCRIPTION CONSISTENT ONLY IF $F = 0$!

$\to$ fake flatness

Thm: fake flatness $\Rightarrow$ locally abelian

Proof: arXiv: 1908.08086 [hep-th]

SOLUTION: adjusted curvatures

$$H_{adj.} := dB + \mu_2(A,B) - \frac{1}{3!}\mu_3(A,A,A) - \kappa(A,F)$$

$$\kappa : L_0 \times L_0 \longrightarrow L_{-1}$$

adjustment datum

(extra structure on L_∞-alg.)

$\rightarrow$ κ NOT ALWAYS EXISTS !

$\rightarrow$ EXAMPLES OF κ: string structures

$\rightarrow$ String 2-bundles

$$\left(\widehat{L_0 G} \xrightarrow{t} P_0 G \,,\, D \right)$$

Kac-Moody central extension

References: arXiv: 1403.7185 [hep-th]

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