3d gravity as a source of integrable systems and hierarchies

Juan Carlos Morales Parra

STAMP Seminar
Heriot-Watt University

April, 2023

General Relativity in 3 dimensions

2 key observations about 3d GR

1. Geometry: The Weyl tensor is identically zero.
2. Einstein's equation: The equation $R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R + \Lambda g_{\mu\nu} = 0$ reduces to $R_{\mu\nu} = 2\Lambda g_{\mu\nu}$.

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Local model spacetimes and isometry groups

Λ	Euclidean	Lorentzian
0	$\mathbf{E}^3 = ISO(3)/SO(3)$	$\mathbf{M}^{2+1} = ISO(2,1)/SO(2,1)$
> 0	$\mathbf{S}^3 = SO(4)/SO(3)$	$\mathbf{dS}^{2+1} = SO(3,1)/SO(2,1)$
< 0	$\mathbf{H}^3 = SO(3,1)/SO(3)$	$\mathbf{AdS}^{2+1} = SO(2,2)/SO(2,1)$

Part I: Integrability of Three Dimensional Gravity Field Equations

I Preliminaries: SL(2, $\mathbb{R}$) solitons and AKNS system

SL(2, $\mathbb{R}$) soliton connection

$$a = Pdx + Qdt$$

with (Lax pair)

$$P = \begin{pmatrix} \xi & p(t,x) \\ q(t,x) & -\xi \end{pmatrix}, \quad Q = \begin{pmatrix} A(p,q,\partial_x p,\partial_x q) & B(p,q,\partial_x p,\partial_x q) \\ C(p,q,\partial_x p,\partial_x q) & -A(p,q,\partial_x p,\partial_x q) \end{pmatrix}$$

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With flat condition

$$0 = F(a) = \frac{\partial P}{\partial t} - \frac{\partial Q}{\partial x} + [P,Q] = 0 \tag{1}$$

I Preliminaries: $SL(2,\mathbb{R})$ solitons and AKNS system

The AKNS equation

Explicity

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is equivalent to the following set of PDE's

$$\begin{aligned} \frac{\partial A}{\partial x} - pC + qB = 0 \\ \frac{\partial q}{\partial t} + \frac{\partial C}{\partial x} + 2qA - 2\xi C = 0 \\ \frac{\partial p}{\partial t} - \frac{\partial B}{\partial x} - 2pA + 2\xi B = 0 \end{aligned} \tag{2}$$

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called the **AKNS (Ablowitz, Kaup, Newell, Segur) equations** [(1974), "The inverse scattering transform-Fourier analysis for nonlinear problems", Studies in Appl. Math., 53 (4): 249–315].

Relation with integrable systems

Polynomial expansion in terms of the spectral parameter

Assume the functions A, B, C are polynomials of the spectral parameter ξ, such that $deg(A) \leq 3$, $deg(B) \leq 3$ and $deg(C) \leq 3$, i.e. $A = \sum_{n=0}^{3} a_i \xi^{3-n}$ (same for B, C).

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Equivalent formulation of AKNS system

$$b_0 = 0, \qquad c_0 = 0$$

$$\frac{\partial a_i}{\partial x} - pc_i + qb_i = 0 \qquad i = 0, 1, 2, 3$$

$$\frac{\partial c_i}{\partial x} + 2qa_i - 2c_{i+1} = 0, \qquad \frac{\partial b_i}{\partial x} + 2pa_i - 2b_{i+1} = 0 \qquad i = 0, 1, 2$$

$$\frac{\partial q}{\partial t} + \frac{\partial c_3}{\partial x} + 2qa_3 = 0, \qquad \frac{\partial p}{\partial t} - \frac{\partial b_3}{\partial x} - 2pa_3 = 0$$

Integrability of AKNS system

Equations for p and q

Seeing the previous as a system for $p(t,x)$ and $q(t,x)$ it reduces to

$$\frac{\partial q}{\partial t} = a_2\left(-\frac{1}{2}\frac{\partial^2 q}{\partial x^2} + q^2 p\right) + a_3\left(-\frac{1}{4}\frac{\partial^3 q}{\partial x^3} + \frac{3}{2}qp\frac{\partial q}{\partial x}\right)$$

$$\frac{\partial p}{\partial t} = a_2\left(\frac{1}{2}\frac{\partial^2 p}{\partial x^2} - qp^2\right) + a_3\left(-\frac{1}{4}\frac{\partial^3 p}{\partial x^3} + \frac{3}{2}qp\frac{\partial p}{\partial x}\right)$$

with a_2 and a_3 constants.

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with a_2 and a_3 constants.

Integrability

Hence the polynomial expansion leads to recognize the connection between the integrability of $SL(2,\mathbb{R})$ solitons and the integrability of:

- Non-linear Schrödinger systems ($a_3 = 0$)
- Modified KdV systems ($a_2 = 0$)

"Ingredients" of the N-P formalism

A.) $SL(2,\mathbb{R})$-valued tetrad 1-form

$$\tilde{\sigma} = \begin{pmatrix} \boldsymbol{n} & -\frac{1}{\sqrt{2}}\boldsymbol{m} \\ -\frac{1}{\sqrt{2}}\boldsymbol{m} & \boldsymbol{\ell} \end{pmatrix} \tag{3}$$

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B.) $\mathfrak{sl}(2,\mathbb{R})$-valued connection 1-form

$$\omega = \begin{pmatrix} \omega_0 & \omega_2 \\ \omega_1 & -\omega_0 \end{pmatrix} \tag{4}$$

with

$$\omega_0 = \frac{1}{2}(-\epsilon'\boldsymbol{\ell} + \epsilon\boldsymbol{n} - \alpha\boldsymbol{m})$$

$$\omega_1 = \frac{1}{\sqrt{2}}(-\tau\boldsymbol{\ell} - \kappa\boldsymbol{n} + \sigma\boldsymbol{m})$$

$$\omega_2 = \frac{1}{\sqrt{2}}(-\kappa'\boldsymbol{\ell} - \tau'\boldsymbol{n} + \sigma'\boldsymbol{m})$$

Equations of the N-P formalism

Spin 1-connection equation

$$d\tilde{\sigma} + \omega\tilde{\sigma} - \tilde{\sigma}\omega^t = 0 \quad (5)$$

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It is equivalent to the following system of equations involving the null-tetrads and the N-P spin coefficients:

$$d\boldsymbol{n} = \epsilon' \boldsymbol{\ell n} - \kappa' \boldsymbol{\ell m} - (\alpha + \tau')\boldsymbol{nm}$$
$$d\boldsymbol{\ell} = -\epsilon \boldsymbol{\ell n} + (\alpha - \tau)\boldsymbol{\ell m} + \kappa' \boldsymbol{nm}$$
$$d\boldsymbol{m} = (\tau' - \tau)\boldsymbol{\ell n} - \sigma' \boldsymbol{\ell m} - \sigma \boldsymbol{nm}$$

Reduction of the 2-form curvature in 3d

Curvature 2-form equation

$$\boldsymbol{R} = d\omega + \omega\omega = \begin{pmatrix} R_0 & R_2 \\ R_1 & -R_0 \end{pmatrix} \tag{6}$$

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y direct computation we get (using the Spin 1-connection equation)

$$R_0 = (2\Phi_{11} + \Lambda)\boldsymbol{\ell n} - \Phi_{12}\boldsymbol{\ell m} + \Phi_{10}\boldsymbol{nm}$$

$$R_1 = -\sqrt{2}\Phi_{01}\boldsymbol{\ell n} + \frac{1}{\sqrt{2}}(\Phi_{02} - 2\Lambda)\boldsymbol{\ell m} - \sqrt{2}\Phi_{00}\boldsymbol{nm}$$

$$R_2 = \sqrt{2}\Phi_{12}\boldsymbol{\ell n} - \sqrt{2}\Phi_{22}\boldsymbol{\ell m} + \frac{1}{\sqrt{2}}(\Phi_{02} - 2\Lambda)\boldsymbol{nm}$$

were Φ_{ij} $(i, j = 0, 1, 2)$ are the trace free Ricci spin coefficients and $\Lambda = -R/6$.

3 dimensional case: Particularity I

$$\boldsymbol{R} = \Lambda \begin{pmatrix} \ell\boldsymbol{n} & -\sqrt{2}\boldsymbol{n}\boldsymbol{m} \\ \sqrt{2}\ell\boldsymbol{m} & -\ell\boldsymbol{n} \end{pmatrix} \tag{7}$$

3 dimensional case: Particularity I

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3 dimensional case: Particularity II

Using $\epsilon = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$, by direct computation we obtain

$$\tilde{\sigma}\epsilon = \begin{pmatrix} \frac{1}{\sqrt{2}}\boldsymbol{m} & \boldsymbol{n} \\ -\boldsymbol{\ell} & -\frac{1}{\sqrt{2}}\boldsymbol{m} \end{pmatrix} \quad \text{and} \quad \tilde{\sigma}\epsilon\tilde{\sigma}\epsilon = \begin{pmatrix} \boldsymbol{\ell n} & -\sqrt{2}\boldsymbol{nm} \\ \sqrt{2}\boldsymbol{\ell m} & -\boldsymbol{\ell n} \end{pmatrix}$$

Combining the "ingredients" of the N-P formalism

Construct the 1-form connection

$$\Gamma_{\pm} = \omega \pm \lambda \tilde{\sigma} \epsilon$$

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Curvatures of $\Gamma_{\pm}$

$$\begin{aligned}\Omega_{\pm} &= d\Gamma_{\pm} + \Gamma_{\pm}\Gamma_{\pm} \\ &= \boldsymbol{R} + \lambda^2\tilde{\sigma}\epsilon\tilde{\sigma}\epsilon \pm \lambda\tilde{\sigma}(\omega^t\epsilon + \epsilon\omega)\end{aligned}$$

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Since $d\epsilon = 0$ and choosing $\lambda^2 = -\Lambda$ then ...

$$\Omega_{\pm} = 0 \tag{8}$$

i.e. $\Gamma_{\pm}$ are flat $\mathfrak{sl}(2,\mathbb{R})$-connections.

Relating $\Gamma_\pm$ with AKNS solitons

Γ and $\mathfrak{sl}(2,\mathbb{R})$ connections

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Comment: Let

$$\Gamma_\pm = \begin{pmatrix} X_\pm & Y_\pm \\ Z_\pm & -X_\pm \end{pmatrix}$$

then the 1-form connections $\tilde{\sigma}$ and ω are recovered by

$$\boldsymbol{m} = \frac{1}{\sqrt{2}\lambda}(X_+ - X_-), \quad \boldsymbol{n} = \frac{1}{2\lambda}(Y_+ - Y_-), \quad \boldsymbol{\ell} = -\frac{1}{2\lambda}(Z_+ - Z_-)$$

Relating $\Gamma_\pm$ with the metric

Let

$$a^{\pm} = P^{\pm} dx + Q^{\pm} dt \quad \text{and} \quad b = \begin{pmatrix} \alpha & \beta \\ \gamma & \sigma \end{pmatrix}$$

where

$$P^{\pm} = \begin{pmatrix} \xi^{\pm} & p^{\pm}(t,x) \\ q^{\pm}(t,x) & -\xi^{\pm} \end{pmatrix}, Q^{\pm} = \begin{pmatrix} A^{\pm} & B^{\pm} \\ C^{\pm} & -A^{\pm} \end{pmatrix} (p^{\pm}, q^{\pm}, \partial_x (p^{\pm}, q^{\pm}))$$

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Then

$$\begin{aligned} X &= [2(\alpha\sigma + \beta\gamma)\xi + \sigma\gamma p - \alpha\beta q]dx + [(\alpha\sigma + \beta\gamma)A + \sigma\gamma B - \alpha\beta C]dt \\ &\qquad + \sigma d\alpha - \beta d\gamma \\ Y &= [4\sigma\beta\xi + \sigma^2 p - \beta^2 q]dx + (2\sigma\beta A + \sigma^2 B - \beta^2 C)dt + \sigma d\beta - \beta d\sigma \\ Z &= (-4\xi\gamma\alpha - \gamma^2 p + \alpha^2 q)dx + [-2\alpha\gamma A - \gamma^2 B + \alpha^2 C]dt - \gamma d\alpha + \alpha d\gamma \end{aligned}$$

Relating $\Gamma_\pm$ with the metric

3d metric in terms of the N-P tetrads

$$g_{\mu\nu} = \ell_\mu n_\nu + \ell_\nu \eta_\mu - m_{mu} m_\nu$$

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Field equations

$$\begin{aligned} \partial_x A^\pm - p^\pm C^\pm + q^\pm B^\pm &= 0 \\ \partial_t q^\pm + \partial_x C^\pm - 2q^\pm A^\pm - 2\xi^\pm C^\pm &= 0 \\ \partial_t p^\pm + \partial_x B^\pm + 2p^\pm A^\pm + 2\xi^\pm B^\pm &= 0 \end{aligned}$$

Showing explicitly how the integrability for the gravitational field equations in three dimensions follows from the integrability of the AKNS system.

Part II: Integrability boundary dynamics of higher spin gravity

II Preliminaries: mBoussinesq hierarchy

1st member of mBoussinesq hierarchy

$$\begin{aligned} \frac{\partial \mathcal{J}}{\partial t} &= \lambda_1 \frac{\partial \mathcal{J}}{\partial \phi} - \lambda_2 \left(2 \frac{\partial (\mathcal{J}\mathcal{U})}{\partial \phi} + \frac{\partial^2 \mathcal{U}}{\partial \phi^2} \right) \\ \frac{\partial \mathcal{U}}{\partial t} &= \lambda_1 \frac{\partial \mathcal{U}}{\partial \phi} + \lambda_2 \left(\frac{\partial (\mathcal{U}^2)}{\partial \phi} - \frac{\partial (\mathcal{J}^2)}{\partial \phi} + \frac{\partial^2 \mathcal{J}}{\partial \phi^2} \right) \end{aligned} \tag{9}$$

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Specific cases

- $\lambda_1 = \lambda_2 = 1$ (mBoussinesq equation).
- $\lambda_1 = 1$, $\lambda_2 = 0$ (2 independent chiral fields).

II Preliminaries: Hamiltonian formalism of mBoussinesq hierarchy

1st Hamiltonian of the hierarchy

$$H_{(1)} = \frac{k}{4\pi} \int d\phi \left[\frac{\lambda_1}{2}(\mathcal{J}^2 + \mathcal{U}^2) + \lambda_2 \left(\frac{1}{3}\mathcal{U}^3 - \mathcal{J}^2\mathcal{U} - \mathcal{J}\frac{\partial U}{\partial \phi} \right) \right] \quad (10)$$

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Poisson bracket

$$\{F,G\} = \frac{4\pi}{k}\int d\phi\left[\frac{\delta F}{\delta\mathcal{J}}\frac{\partial}{\partial_\phi}\left(\frac{\delta G}{\delta\mathcal{J}}\right)+\frac{\delta F}{\delta\mathcal{U}}\frac{\partial}{\partial_\phi}\left(\frac{\delta G}{\delta\mathcal{U}}\right)\right] \tag{11}$$

II Preliminaries: Hamiltonian formalism of mBoussinesq hierarchy

Dynamics

By direct computation we have

$$\{\mathcal{J}, H_{(1)}\}(t,\phi) = \frac{4pi}{k}\int d\phi' \delta(\phi-\phi')\frac{\partial}{\partial\phi}\left(\frac{\delta H_{(1)}}{\partial \mathcal{J}}\right)$$
$$= \lambda_1 \frac{\partial \mathcal{J}}{\partial \phi} - \lambda_2\left(2\frac{\partial(\mathcal{J}\mathcal{U})}{\partial\phi} + \frac{\partial^2\mathcal{U}}{\partial\phi^2}\right)$$
$$= \frac{\partial \mathcal{J}}{\partial t}$$

and similarly

$$\{\mathcal{U}, H_{(1)}\}(t,\phi) = \frac{\partial \mathcal{U}}{\partial t}$$

2 Comments

1st comment: Matrix representation of the 1st member of mBoussinesq hierarchy

$$\begin{pmatrix} \frac{\partial \mathcal{J}}{\partial t} \\ \frac{\partial \mathcal{U}}{\partial t} \end{pmatrix} = \mathcal{D} \begin{pmatrix} \frac{\delta H_{(1)}}{\delta \mathcal{J}} \\ \frac{\delta H_{(1)}}{\delta \mathcal{U}} \end{pmatrix} \qquad \text{with} \quad \mathcal{D} = \frac{4\pi}{k} \begin{pmatrix} \partial_\phi & 0 \\ 0 & \partial_\phi \end{pmatrix}$$

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2nd comment: Relation with higher spin 3d gravity

$$\{\mathcal{J}(\phi), \mathcal{J}(\phi')\} = \frac{4\pi}{k} \partial_\phi \delta(\phi - \phi')$$

$$\{\mathcal{U}(\phi), \mathcal{U}(\phi')\} = \frac{4\pi}{k} \partial_\phi \delta(\phi - \phi')$$

Once appropriate boundary conditgions are imposed this Poisson algebra is obtained from Dirac brackets in higher spin theory!

II Preliminaries: Integrability and hierarchy

bi-Hamiltonian

Then integrability and existence of hierarchy follows from the fact the system is bi-Hamiltonian:

$$H_{(0)} = \frac{k}{4\pi}\int d\phi(\lambda_1\mathcal{J} + \lambda_2\mathcal{U})$$

$$\{F, G\}_2 = \int d\phi \begin{pmatrix} \frac{\delta F}{\delta \mathcal{J}} & \frac{\delta F}{\delta \mathcal{U}} \end{pmatrix} \mathcal{D}_{(2)} \begin{pmatrix} \frac{\delta G}{\delta \mathcal{J}} \\ \frac{\delta G}{\delta \mathcal{U}} \end{pmatrix}$$

where

$$\mathcal{D}_{(2)} = \mathcal{D}M^\dagger\mathcal{O}M\mathcal{D} \quad \text{with} \quad \mathcal{O} = \frac{2k}{\pi}\begin{pmatrix} 0 & \partial_\phi^{-1} \\ \partial_\phi^{-1} & 0 \end{pmatrix}$$

and

$$M = \begin{pmatrix} \mathcal{J} + \partial_\phi & \mathcal{U} \\ -2\mathcal{J}\mathcal{U} - \frac{1}{2}\mathcal{U}\partial_\phi - \frac{3}{2}\frac{\partial\mathcal{U}}{\partial\phi} & \mathcal{U}^2 - \mathcal{J}^2 - \frac{1}{2}\frac{\partial\mathcal{J}}{\partial\phi} - \frac{3}{2}\mathcal{J}\partial_\phi - \frac{1}{2}\partial_\phi^2 \end{pmatrix}$$

Integrablity: Conserved charges by recursion

Gelfand-Dickey polynomials

Infinite number of conserved charges in involution by recursion

$$\mathcal{R}_{(n+1)} = \mathcal{D}^{-1}\mathcal{D}_{(2)}\mathcal{R}_{(n)}, \quad \text{where} \quad \mathcal{R}_{(n)} = \begin{pmatrix} \frac{\delta H_{(n)}}{\delta \mathcal{J}} \\ \frac{\delta H_{(n)}}{\delta \mathcal{U}} \end{pmatrix} \tag{12}$$

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Charges in involution

Decomposing $H_{(n)} = \lambda_1 H^1_{(n)} + \lambda_2 H^2_{(n)}$ then

$$\left\{H^i_{(n)}, H^j_{(m)}\right\} = \left\{H^i_{(n)}, H^j_{(m)}\right\}_2 = 0 \quad \text{for} \quad i,j = 1,2 \text{ and } n,m \in \mathbb{Z}_{\geq 0}$$

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Hierarchy of integrable equations labeled by $k \in \mathbb{Z}^+$

$$\frac{\partial \mathcal{J}}{\partial t} = \{\mathcal{J}, H_{(k)}\} = \{\mathcal{J}, H_{(K-1)}\}_2, \quad \frac{\partial \mathcal{U}}{\partial t} = \{\mathcal{U}, H_{(k)}\} = \{\mathcal{U}, H_{(K-1)}\}_2$$

Review: Chern-Simons formulation of spin-3 gravity in AdS_3

Chern-Simons for $\mathrm{SL}(3,\mathbb{R}) \times \mathrm{SL}(3,\mathbb{R})$

$$I = I_{CS}[A^+] - I_{CS}[A^-], \quad \text{where} \quad I_{CS}[A] = \frac{k}{16\pi}\int_{\mathcal{M}} \mathrm{tr}\left(AdA + \frac{2}{3}A^3\right)$$

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Equations of motion (Flat connections)

$$F^{\pm} = dA^{\pm} + {A^{\pm}}^2 = 0$$

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Equations of motion (Flat connections)

$$F^{\pm} = dA^{\pm} + {A^{\pm}}^2 = 0$$

Recovering metric and spin fields

With the generalized dreiben $e := \frac{\ell}{2}(A^+ - A^-)$,

$$g_{\mu\nu} = \frac{1}{2}\mathrm{tr}(e_\mu e_\nu), \qquad \varphi_{\mu\nu\rho} = \frac{1}{3!}\mathrm{tr}(e_{(\mu} e_\nu e_{\rho)})$$

Review: Asymptotic behaviour of fields

Diagonal gauge

$$A = b^{-1}(d + a)b$$

where $b = b(r)$ and

$$a = (\mathcal{J}\, d\phi + \zeta dt)L_0 + \frac{\sqrt{3}}{2}(\mathcal{U}\, d\phi + \zeta_{\mathcal{U}} dt)W_0$$

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Boundary conditions

- $\zeta, \zeta_{\mathcal{U}}$ **fixed** at the boundary: [D. Grumiller, A. Pérez, S. Prohazka, D. Tempo and R. Troncoso, **Higher Spin Black Holes with Soft Hair**, JHEP 10 (2016) 119].

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- $\zeta, \zeta_{\mathcal{U}}$ **precise functional dependence** on $\mathcal{J}, \mathcal{U}, \partial(\mathcal{J}, \mathcal{U})$: [E.Ojeda and A. Pérez, **Boundary conditions for General Relativity in three-dimensional spacetimes, integrable systems and the KdV/mKdV hierarchies**, JHEP 08 (2019) 079]

Boundary conditions for spin-3 gravity and mBoussinesq hierarchy

Compatibility of boundary conditions with action principle

$$I_{\mathrm{Can}}[A] = -\frac{k}{16\pi}\int dtdx^2 \epsilon^{ij}\mathrm{tr}(A_i\partial_t A_j - A_t F_{ij}) + B_\infty$$

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Choice of boundary Lagrange multipliers [E.Ojeda, et al.]

$$\zeta = \frac{4\pi}{k}\frac{\delta H_{(k)}}{\delta\mathcal{J}}, \qquad \zeta_{\mathcal{U}} = \frac{4\pi}{k}\frac{\delta H_{(k)}}{\delta\mathcal{U}}$$

The mBoussinesq hierarchy from spin-3 gravity

Equations of motion

The field equations in canonical higher spin theory (vanishing of field strenght) are

$$\frac{\partial \mathcal{J}}{\partial t} = \frac{\partial \zeta}{\partial \phi}, \qquad \frac{\partial \mathcal{U}}{\partial t} = \frac{\partial \zeta_{\mathcal{U}}}{\partial \phi}$$

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So for our choice of Lagrange multipliers...

$$\begin{pmatrix} \frac{\partial \mathcal{J}}{\partial t} \\ \frac{\partial \mathcal{U}}{\partial t} \end{pmatrix} = \mathcal{D} \begin{pmatrix} \frac{\delta H_{(k)}}{\delta \mathcal{J}} \\ \frac{\delta H_{(k)}}{\delta \mathcal{U}} \end{pmatrix} = \begin{pmatrix} \{\mathcal{J}, H_{(k)}\} \\ \{\mathcal{U}, H_{(k)}\} \end{pmatrix}$$

i.e. the equations of motion correspond to the kth element in the mBoussinesq hierarchy.

Asymptotic symmetries

Gauge transformations preserving asymptotic form of A

$$\delta a = d\lambda + [a, \lambda], \quad \text{with } \lambda = \eta L_0 + \frac{\sqrt{3}}{2}\eta_{\mathcal{U}} W_0$$

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Transformation rules

- **Angular components:** $\delta\mathcal{J} = \frac{\partial\eta}{\partial\phi}$ and $\delta\mathcal{U} = \frac{\partial\eta_{\mathcal{U}}}{\partial\phi}$
- **Temporal components:** $\delta\zeta = \frac{\partial\eta}{\partial t}$ and $\delta\zeta_{\mathcal{U}} = \frac{\partial\eta_{\mathcal{U}}}{\partial t}$

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Dirac brackets for $\mathcal{J}$ and $\mathcal{U}$ [Using Regge-Teitelboim method]

$$\{\mathcal{J}(\phi), \mathcal{J}(\phi')\}^{\star} = \frac{4\pi}{k}\partial_{\phi}(\phi - \phi')$$

$$\{\mathcal{U}(\phi), \mathcal{U}(\phi')\}^{\star} = \frac{4\pi}{k}\partial_{\phi}(\phi - \phi')$$

as expected!

Conserved Charges

The consistency transformations of the temporal components imply

$$\begin{pmatrix} \partial_t \eta(t,\theta) \\ \partial_t \eta_{\mathcal{U}}(t,\theta) \end{pmatrix} = \int d\phi \,\mathsf{Hess}_{\mathcal{J},\mathcal{U}}[H_{(k)}](t,\theta;t,\phi)\mathcal{D}\begin{pmatrix} \eta \\ \eta_{\mathcal{U}} \end{pmatrix}$$

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So, as a consequence of integrability ...

$$\begin{pmatrix} \eta \\ \eta_{\mathcal{U}} \end{pmatrix} = \frac{4\pi}{k}\sum_{n=0}^{\infty}\alpha_{(n)}\begin{pmatrix} \frac{\delta H_{(n)}}{\delta \mathcal{J}} \\ \frac{\delta H_{(n)}}{\delta \mathcal{U}} \end{pmatrix}$$

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Conserved charges=Linear combination of Hamiltonians!!!

$$Q = \sum_{n=0}^{\infty}\alpha_{(n)}H_{(n)}$$

References

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Thank you for your attention!

Questions?