Lecture 2: Clifford algebras: the classification

The small part of ignorance that we arrange and classify we give the name of knowledge.

—Ambrose Bierce

In this section we will classify finite-dimensional real and complex Clifford algebras. Useful references are [ABS64] (which treats only the positive- and negative-definite cases, albeit lucidly) and [LM89, Har90].

2.1 A less-than-useful classification

We start with a result which is interesting in its own right, but perhaps not as useful as it may appear at first.

Given two quadratic vector spaces (V, Q_V) and (W, Q_W), we can form their orthogonal direct sum $(V \oplus W, Q_V \oplus Q_W)$. (Notice that although the direct sum is the coproduct in the category of vector spaces, it is not the coproduct in **QVec**.) One natural question is whether the Clifford algebra $C\ell(V \oplus W, Q_V \oplus Q_W)$ of the orthogonal direct sum is related to the Clifford algebras $C\ell(V, Q_V)$ and $C\ell(W, Q_W)$ of its summands. The answer is very simple, which shows why one should take the $\mathbb{Z}_2$-grading of the Clifford algebra very seriously!

Proposition 2.1. *Let (V, Q_V) and (W, Q_W) be quadratic vector spaces and let $(V \oplus W, Q_V \oplus Q_W)$ be their orthogonal direct sum. Then there is an isomorphism of $\mathbb{Z}_2$-graded associative algebras:*

$$C\ell(V \oplus W, Q_V \oplus Q_W) \cong C\ell(V, Q_V) \hat{\otimes} C\ell(W, Q_W) ,$$

where $\hat{\otimes}$ denotes the $\mathbb{Z}_2$-graded tensor product.

Tensor product of algebras

If A and B are associative algebras over the same ground field, their vector space tensor product $A \otimes B$ becomes an algebra by

$$(a_1 \otimes b_1)(a_2 \otimes b_2) = a_1 a_2 \otimes b_1 b_2$$

and extending bilinearly. However when $A = A_0 \oplus A_1$ and $B = B_0 \oplus B_1$ are themselves $\mathbb{Z}_2$-graded, we can define on the $\mathbb{Z}_2$-graded tensor product $A \hat{\otimes} B$, with $(A \hat{\otimes} B)_0 = (A_0 \otimes B_0) \oplus (A_1 \otimes B_1)$ and $(A \hat{\otimes} B)_1 = (A_0 \otimes B_1) \oplus (A_1 \otimes B_0)$, the following associative multiplication

$$(a_1 \otimes b_1)(a_2 \otimes b_2) = (-1)^{|a_2||b_1|} a_1 a_2 \otimes b_1 b_2 , \tag{42}$$

where a_i and b_i are homogeneous elements of parity $|a_i|$, $|b_i|$, respectively, and extending it bilinearly to all of $A \hat{\otimes} B$.

Proof. Define a linear map

$$\phi : V \oplus W \longrightarrow C\ell(V, Q_V) \hat{\otimes} C\ell(W, Q_W) \tag{43}$$

by $\phi(v + w) = v \otimes \mathbf{1} + \mathbf{1} \otimes w$, where $v \in V$ and $w \in W$ and we are identifying V with its image in $C\ell(V, Q_V)$ and similarly for W. One checks that this map is Clifford *precisely* because of the sign in the definition (42) of the multiplication in the $\mathbb{Z}_2$-graded tensor product. Indeed,

$$\begin{aligned}
\phi(v + w)^2 &= (v \otimes \mathbf{1} + \mathbf{1} \otimes w)^2 \\
&= v^2 \otimes \mathbf{1} + v \otimes w + (-1)^{|v||w|} v \otimes w + \mathbf{1} \otimes w^2 \\
&= -(Q(v) + Q(w)) \mathbf{1} \otimes \mathbf{1} .
\end{aligned}$$

By universality it extends to a homomorphism of $\mathbb{Z}_2$-graded associative algebras

$$\Phi : C\ell(V \oplus W, Q_V \oplus Q_W) \longrightarrow C\ell(V, Q_V) \hat{\otimes} C\ell(W, Q_W) ,$$

which is injective since it is one-to-one on generators and is surjective because the image contains $C\ell(V, Q_V)\hat{\otimes}1$ and $1\hat{\otimes}C\ell(W, Q_W)$, which together generate $C\ell(V, Q_V)\hat{\otimes}C\ell(W, Q_W)$. ☐

Now every (finite-dimensional) real quadratic vector space (V, Q) is isomorphic to an orthogonal direct sum

$$\underbrace{\mathbb{R}(0) \oplus \cdots \oplus \mathbb{R}(0)}_{r} \oplus \underbrace{\mathbb{R}(+) \oplus \cdots \oplus \mathbb{R}(+)}_{s} \oplus \underbrace{\mathbb{R}(-) \oplus \cdots \oplus \mathbb{R}(-)}_{t} \tag{44}$$

where $\mathbb{R}(0)$ is the one-dimensional vector space with zero quadratic form, whereas $\mathbb{R}(\pm)$ is the one dimensional vector space with quadratic form $Q(1) = \pm 1$. It then follows from the above proposition that

$$C\ell(V, Q) \cong \Lambda\mathbb{R}^r \hat{\otimes} \underbrace{C\ell(1,0)\hat{\otimes}\cdots\hat{\otimes}C\ell(1,0)}_{s} \hat{\otimes} \underbrace{C\ell(0,1)\hat{\otimes}\cdots\hat{\otimes}C\ell(0,1)}_{t} , \tag{45}$$

where we have used that the Clifford algebra associated to the zero quadratic form is the exterior algebra. Using that $C\ell(1,0) \cong \mathbb{C}$ and $C\ell(0,1) \cong \mathbb{R}\oplus\mathbb{R}$, the above result determines in principle all the finite-dimensional real Clifford algebras as $\mathbb{Z}_2$-graded associative algebras. This is nice, but one can do much better and actually identify the Clifford algebras in terms of the matrix algebras $\mathbb{K}(n)$, for $\mathbb{K} = \mathbb{R}, \mathbb{C}, \mathbb{H}$ as we started doing in the first lecture.

2.2 Complex Clifford algebras

Before presenting the classification, let us consider the complex Clifford algebras. Since a complex quadratic form has no signature, every (finite-dimensional) complex quadratic vector space is isomorphic to an orthogonal direct sum

$$\underbrace{\mathbb{C}(0) \oplus \cdots \oplus \mathbb{C}(0)}_{r} \oplus \underbrace{\mathbb{C}(+) \oplus \cdots \oplus \mathbb{C}(+)}_{n} , \tag{46}$$

where $\mathbb{C}(0)$ is the one-dimensional complex vector spaces with zero quadratic form and $\mathbb{C}(+)$ the one-dimensional complex vector space with $Q(1) = 1$. Proposition 2.1 then says that the corresponding complex Clifford algebra is isomorphic to

$$\Lambda\mathbb{C}^r \hat{\otimes} \underbrace{\mathbb{C}\ell(1)\hat{\otimes}\cdots\hat{\otimes}\mathbb{C}\ell(1)}_{n} , \tag{47}$$

where $\mathbb{C}\ell(1)$ denotes the Clifford algebra $C\ell(\mathbb{C}(+))$. To identify it, notice that $\mathbb{C}\ell(1)$ is the complex associative algebra generated by e obeying $e^2 = -\mathbf{1}$. This means that $(ie)^2 = \mathbf{1}$ and we define complementary projectors $p_\pm = \frac{1}{2}(\mathbf{1} \pm ie)$, which induce an isomorphism $\mathbb{C}\ell(1) \cong \mathbb{C}\oplus\mathbb{C}$ with $zp_+ + wp_- \leftrightarrow (z, w)$.

One can complexify real quadratic vector spaces as follows. Let (V, Q) be a real quadratic vector space and let $V_\mathbb{C} = V \otimes_\mathbb{R} \mathbb{C}$ be its complexification. We extend Q complex linearly to a quadratic form $Q_\mathbb{C}$ defined by $Q_\mathbb{C}(v \otimes z) = z^2 Q(v)$. This turns the pair $(V_\mathbb{C}, Q_\mathbb{C})$ into a complex quadratic vector space. It is natural to ask whether $C\ell(V, Q)$ and $C\ell(V_\mathbb{C}, Q_\mathbb{C})$ are related and the answer could not be nicer.

Proposition 2.2. *The Clifford functor* $C\ell$ *commutes with complexification; that is,*

$$C\ell(V_\mathbb{C}, Q_\mathbb{C}) \cong C\ell(V, Q) \otimes_\mathbb{R} \mathbb{C} .$$

Proof. The map $V \times \mathbb{C} \to C\ell(V, Q) \otimes_\mathbb{R} \mathbb{C}$ defined by $\phi(v, z) = v \otimes z$ is real bilinear, whence it defines an $\mathbb{R}$-linear map

$$V \otimes_\mathbb{R} \mathbb{C} \xrightarrow{\phi} C\ell(V, Q) \otimes_\mathbb{R} \mathbb{C} .$$

Since $\phi(v \otimes z) = v \otimes z = (v \otimes 1)z = \phi(v \otimes 1)z$, we see that it is also $\mathbb{C}$-linear. Also since $\phi(v \otimes 1)^2 = (v \otimes 1)^2 = -Q(v)(\mathbf{1} \otimes 1)$, we see that ϕ is Clifford, whence it extends uniquely to a homomorphism of complex algebras

$$\Phi : C\ell(V_{\mathbb{C}}, Q_{\mathbb{C}}) \longrightarrow C\ell(V, Q) \otimes_{\mathbb{R}} \mathbb{C} .$$

It is clearly injective on generators and either by counting dimension or by observing that the image of Φ contains $C\ell(V, Q) \otimes 1$ and $\mathbf{1} \otimes \mathbb{C}$ and these generate the right-hand side, we conclude that Φ is an isomorphism. □

As a corollary we have that upon complexification we lose the information about the signature:

$$C\ell(s,t) \otimes_{\mathbb{R}} \mathbb{C} \cong \mathbb{C}\ell(s+t) . \tag{48}$$

The absence of signature for a complex quadratic vector space means that the complex Clifford algebras have a much simpler structure than the real Clifford algebras.

2.3 Filling in the Clifford chessboard

From now on we will restrict attention to the case of nondegenerate quadratic forms, whence any real quadratic vector space is isomorphic to $\mathbb{R}^{s,t}$ for some s, t. We would like to identify the Clifford algebras $C\ell(s,t)$ for all $s, t \geq 0$. In the last lecture we already filled in a corner of the table of Clifford algebras:

$\mathbb{R}(2)$							
$\mathbb{R} \oplus \mathbb{R}$	$\mathbb{R}(2)$						
$\mathbb{R}$	$\mathbb{C}$	$\mathbb{H}$					

$\xrightarrow[s]{\qquad}$

This corner of the table is enough to fill in the rest of the table, thanks to the following isomorphisms.

Theorem 2.3. *For all $n, s, t \geq 0$ we have the following isomorphisms*

$$\begin{aligned} C\ell(n,0) \otimes C\ell(0,2) &\cong C\ell(0,n+2) \\ C\ell(0,n) \otimes C\ell(2,0) &\cong C\ell(n+2,0) \\ C\ell(s,t) \otimes C\ell(1,1) &\cong C\ell(s+1,t+1) \end{aligned} \tag{49}$$

with $\otimes$ the ungraded tensor product.

Proof. As the three cases are very similar, we shall prove the second equation in (49) and leave the other two as exercises for the reader.

Let us write $\mathbb{R}^{n+2,0} = \mathbb{R}^{n,0} \oplus \mathbb{R}^{2,0}$. Let e_1, e_2 be an orthonormal basis for $\mathbb{R}^{2,0}$ and let us denote by the same symbols their image in $C\ell(2,0)$. This means that $e_1^2 = e_2^2 = -\mathbf{1}$ and $e_1 e_2 = -e_2 e_1$. The element $e_1 e_2 \in C\ell(2,0)$ satisfies the following easily verifiable identities: $(e_1 e_2)^2 = -\mathbf{1}$, $e_1 e_2 e_i = -e_i e_1 e_2$ for $i = 1, 2$. Let us define a linear map

$$\phi : \mathbb{R}^{n+2,0} \longrightarrow C\ell(0,n) \otimes C\ell(2,0)$$

by

$$\phi(x) = x \otimes e_1 e_2 \qquad \text{and} \qquad \phi(e_i) = \mathbf{1} \otimes e_i ,$$

for $x \in \mathbb{R}^{n,0}$. This map is Clifford by virtue of the identities satisfied by $e_1 e_2$; indeed,

$$\begin{aligned}
\phi(x + \lambda e_1 + \mu e_2)^2 &= (x \otimes e_1 e_2 + \lambda \mathbf{1} \otimes e_1 + \mu \mathbf{1} \otimes e_2)^2 \\
&= -x^2 \otimes \mathbf{1} - \lambda^2 \mathbf{1} \otimes \mathbf{1} - \mu^2 \mathbf{1} \otimes \mathbf{1} \\
&\quad + \lambda x \otimes (e_1 e_1 e_2 + e_1 e_2 e_1) + \mu x \otimes (e_2 e_1 e_2 + e_1 e_2 e_2) + \lambda\mu \mathbf{1} \otimes (e_1 e_2 + e_2 e_1) \\
&= -(x^2 + \lambda^2 + \mu^2) \mathbf{1} \otimes \mathbf{1} \\
&= -Q(x + \lambda e_1 + \mu e_2) \mathbf{1} \otimes \mathbf{1} .
\end{aligned}$$

Hence ϕ extends uniquely to an algebra homomorphism $\Phi : C\ell(n+2,0) \to C\ell(0,n) \otimes C\ell(2,0)$ which is injective on generators and by dimension must be an isomorphism. □

Notice that the first two isomorphisms in (49) allows us to fill the left column and the bottom row in the table, whereas the last isomorphism allows us to move diagonally. Since any square in the table lies on some diagonal, it can in principle be determined by using the isomorphisms. Moving diagonally (one step) is the same as tensoring with $C\ell(1,1) \cong \mathbb{R}(2)$. To apply this we need to make use of the following standard isomorphism of matrix algebras.

Lemma 2.4. *Let $\mathbb{K}$ stand for any of $\mathbb{R}$, $\mathbb{C}$ and $\mathbb{H}$ and let $\mathbb{K}(n)$ denote the* real *algebra of $n \times n$ matrices with entries in $\mathbb{K}$. Then we have the following isomorphisms of real associative algebras:*

$$\mathbb{K}(m) \otimes_{\mathbb{R}} \mathbb{R}(n) \cong \mathbb{K}(mn) . \tag{50}$$

This already allows us to fill in five of the diagonals in the table:

					$\mathbb{R}(64)$	$\mathbb{R}(64) \oplus \mathbb{R}(64)$	$\mathbb{R}(128)$
				$\mathbb{R}(32)$	$\mathbb{R}(32) \oplus \mathbb{R}(32)$	$\mathbb{R}(64)$	$\mathbb{C}(64)$
			$\mathbb{R}(16)$	$\mathbb{R}(16) \oplus \mathbb{R}(16)$	$\mathbb{R}(32)$	$\mathbb{C}(32)$	$\mathbb{H}(32)$
		$\mathbb{R}(8)$	$\mathbb{R}(8) \oplus \mathbb{R}(8)$	$\mathbb{R}(16)$	$\mathbb{C}(16)$	$\mathbb{H}(16)$	
	$\mathbb{R}(4)$	$\mathbb{R}(4) \oplus \mathbb{R}(4)$	$\mathbb{R}(8)$	$\mathbb{C}(8)$	$\mathbb{H}(8)$		
$\mathbb{R}(2)$	$\mathbb{R}(2) \oplus \mathbb{R}(2)$	$\mathbb{R}(4)$	$\mathbb{C}(4)$	$\mathbb{H}(4)$			
$\mathbb{R} \oplus \mathbb{R}$	$\mathbb{R}(2)$	$\mathbb{C}(2)$	$\mathbb{H}(2)$				
$\mathbb{R}$	$\mathbb{C}$	$\mathbb{H}$					

s

To continue it is necessary to extend the bottom row and the left column. For example, let us continue with the bottom row. From the second of the isomorphisms in (49), we have

$$C\ell(3,0) \cong C\ell(0,1) \otimes C\ell(2,0) \cong (\mathbb{R} \oplus \mathbb{R}) \otimes \mathbb{H} = \mathbb{H} \oplus \mathbb{H} ,$$

where we have used the distributivity of $\otimes$ over $\oplus$. In the same way we obtain

$$C\ell(4,0) \cong C\ell(0,2) \otimes C\ell(2,0) \cong \mathbb{R}(2) \otimes \mathbb{H} = \mathbb{H}(2) .$$

We cannot continue along the bottom row without first extending the left column. Using the first of the isomorphisms in (49), we find

$$C\ell(0,3) \cong C\ell(1,0) \otimes C\ell(0,2) \cong \mathbb{C} \otimes \mathbb{R}(2) \cong \mathbb{C}(2),$$

whereas

$$C\ell(0,4) \cong C\ell(2,0) \otimes C\ell(0,2) \cong \mathbb{H} \otimes \mathbb{R}(2) \cong \mathbb{H}(2),$$

$$C\ell(0,5) \cong C\ell(3,0) \otimes C\ell(0,2) \cong (\mathbb{H} \oplus \mathbb{H}) \otimes \mathbb{R}(2) \cong \mathbb{H}(2) \oplus \mathbb{H}(2),$$

and

$$C\ell(0,6) \cong C\ell(4,0) \otimes C\ell(0,2) \cong \mathbb{H}(2) \otimes \mathbb{R}(2) \cong \mathbb{H}(4) .$$

This allows us to fill in six more diagonals in the table!

	$\mathbb{H}(8)$	$\mathbb{H}(8)\oplus\mathbb{H}(8)$	$\mathbb{H}(16)$	$\mathbb{C}(32)$	$\mathbb{R}(64)$	$\mathbb{R}(64)\oplus\mathbb{R}(64)$	$\mathbb{R}(128)$
$\mathbb{H}(4)$	$\mathbb{H}(4)\oplus\mathbb{H}(4)$	$\mathbb{H}(8)$	$\mathbb{C}(16)$	$\mathbb{R}(32)$	$\mathbb{R}(32)\oplus\mathbb{R}(32)$	$\mathbb{R}(64)$	$\mathbb{C}(64)$
$\mathbb{H}(2)\oplus\mathbb{H}(2)$	$\mathbb{H}(4)$	$\mathbb{C}(8)$	$\mathbb{R}(16)$	$\mathbb{R}(16)\oplus\mathbb{R}(16)$	$\mathbb{R}(32)$	$\mathbb{C}(32)$	$\mathbb{H}(32)$
$\mathbb{H}(2)$	$\mathbb{C}(4)$	$\mathbb{R}(8)$	$\mathbb{R}(8)\oplus\mathbb{R}(8)$	$\mathbb{R}(16)$	$\mathbb{C}(16)$	$\mathbb{H}(16)$	$\mathbb{H}(16)\oplus\mathbb{H}(16)$
$\mathbb{C}(2)$	$\mathbb{R}(4)$	$\mathbb{R}(4)\oplus\mathbb{R}(4)$	$\mathbb{R}(8)$	$\mathbb{C}(8)$	$\mathbb{H}(8)$	$\mathbb{H}(8)\oplus\mathbb{H}(8)$	$\mathbb{H}(16)$
$\mathbb{R}(2)$	$\mathbb{R}(2)\oplus\mathbb{R}(2)$	$\mathbb{R}(4)$	$\mathbb{C}(4)$	$\mathbb{H}(4)$	$\mathbb{H}(4)\oplus\mathbb{H}(4)$	$\mathbb{H}(8)$	
$\mathbb{R}\oplus\mathbb{R}$	$\mathbb{R}(2)$	$\mathbb{C}(2)$	$\mathbb{H}(2)$	$\mathbb{H}(2)\oplus\mathbb{H}(2)$	$\mathbb{H}(4)$		
$\mathbb{R}$	$\mathbb{C}$	$\mathbb{H}$	$\mathbb{H}\oplus\mathbb{H}$	$\mathbb{H}(2)$			

$\xrightarrow[s]{}$

To continue we have to determine $C\ell(5,0)$. From the second of the isomorphisms in (49), we find

$$C\ell(5,0) \cong C\ell(0,3) \otimes C\ell(2,0) \cong \mathbb{C}(2) \otimes \mathbb{H} \cong ?$$

To answer the question we need the following result.

Lemma 2.5. *The following are isomorphisms of real associative algebras:*

1. $\mathbb{C} \otimes_{\mathbb{R}} \mathbb{H} \cong \mathbb{C}(2)$
2. $\mathbb{H} \otimes_{\mathbb{R}} \mathbb{H} \cong \mathbb{R}(4)$
3. $\mathbb{C} \otimes_{\mathbb{R}} \mathbb{C} \cong \mathbb{C} \oplus \mathbb{C}$

Proof. 1. To prove the first isomorphism, let us give $\mathbb{H}$ the structure of a complex vector space by left multiplication by the complex subalgebra of $\mathbb{H}$ generated by i, say. Then we construct a real bilinear map

$$\phi : \mathbb{C} \times \mathbb{H} \longrightarrow \operatorname{End}_{\mathbb{C}}(\mathbb{H}) \quad \text{by} \quad \phi(z,q)x = zx\overline{q},$$

for all $z \in \mathbb{C}$ and $x, q \in \mathbb{H}$. By universality of the tensor product, it defines a real linear map

$$\Phi : \mathbb{C} \otimes_{\mathbb{R}} \mathbb{H} \longrightarrow \operatorname{End}_{\mathbb{C}}(\mathbb{H}) \quad \text{by} \quad \Phi(z \otimes q) = \phi(z,q).$$

We check that Φ is a homomorphism of real algebras:

$$\Phi(z_1 \otimes q_1)\Phi(z_2 \otimes q_1)x = z_1(z_2x\overline{q_2})\overline{q_1} = (z_1z_2)x\overline{q_1q_2} = \Phi(z_1z_2 \otimes q_1q_2)x\,.$$

It is clearly injective because $\mathbb{C}$ and $\mathbb{H}$ are division algebras and counting dimension ($\dim_{\mathbb{R}} = 8$) we see that Φ must be an isomorphism. Now $\mathbb{H} \cong \mathbb{C}^2$ as a complex vector space, whence $\operatorname{End}_{\mathbb{C}}(\mathbb{H}) \cong \mathbb{C}(2)$.

2. This is proved in a very similar manner to the first isomorphism. Namely we define a real bilinear map

$$\phi : \mathbb{H} \times \mathbb{H} \longrightarrow \operatorname{End}_{\mathbb{R}}(\mathbb{H}) \quad \text{by} \quad \phi(q_1,q_2)x = q_1x\overline{q_2},$$

for all $q_i, x \in \mathbb{H}$, which by universality of the tensor product induces a real linear map

$$\Phi : \mathbb{H} \otimes_{\mathbb{R}} \mathbb{H} \longrightarrow \operatorname{End}_{\mathbb{R}}(\mathbb{H}) \quad \text{by} \quad \Phi(q_1 \otimes q_2) = \phi(q_1,q_2).$$

It is clear that it is injective and counting dimension ($\dim_{\mathbb{R}} = 16$), it is an isomorphism of real vector spaces, but again one checks that Φ is an algebra morphism:

$$\Phi(q_1 \otimes q_2)\Phi(q_1' \otimes q_2')x = q_1(q_1'x\overline{q_2'})\overline{q_2} = (q_1q_1')x\overline{q_2q_2'} = \Phi(q_1q_1' \otimes q_2q_2')x\,.$$

3. This is even easier. Notice that the element $i \otimes i \in \mathbb{C} \otimes_{\mathbb{R}} \mathbb{C}$ squares to the identity, whence we can form complementary projectors $p_\pm = \frac{1}{2}(1 \otimes 1 \pm i \otimes i)$ whose images are commuting subalgebras isomorphic to $\mathbb{C}$. Explicitly, the isomorphism $\mathbb{C} \oplus \mathbb{C} \to \mathbb{C} \otimes_{\mathbb{R}} \mathbb{C}$ is given by $(z_1, z_2) \mapsto z_1p_+ + z_2p_-$.

$\square$

It follows at once from the first of these isomorphisms, that

$$\mathbb{C}(2) \otimes \mathbb{H} \cong \mathbb{R}(2) \otimes \mathbb{C} \otimes \mathbb{H} \cong \mathbb{R}(2) \otimes \mathbb{C}(2) \cong \mathbb{C}(4) \ ,$$

whence $C\ell(5,0) \cong \mathbb{C}(4)$, and hence

$$C\ell(0,7) \cong C\ell(5,0) \otimes C\ell(0,2) \cong \mathbb{C}(4) \otimes \mathbb{R}(2) \cong \mathbb{C}(8) \ .$$

In the same way one can show that $C\ell(6,0) \cong \mathbb{R}(8)$ and $C\ell(7,0) \cong \mathbb{R}(8) \oplus \mathbb{R}(8)$, which allows us to complete the first 8×8 corner of the table:

$\mathbb{C}(8)$	$\mathbb{H}(8)$	$\mathbb{H}(8) \oplus \mathbb{H}(8)$	$\mathbb{H}(16)$	$\mathbb{C}(32)$	$\mathbb{R}(64)$	$\mathbb{R}(64) \oplus \mathbb{R}(64)$	$\mathbb{R}(128)$
$\mathbb{H}(4)$	$\mathbb{H}(4) \oplus \mathbb{H}(4)$	$\mathbb{H}(8)$	$\mathbb{C}(16)$	$\mathbb{R}(32)$	$\mathbb{R}(32) \oplus \mathbb{R}(32)$	$\mathbb{R}(64)$	$\mathbb{C}(64)$
$\mathbb{H}(2) \oplus \mathbb{H}(2)$	$\mathbb{H}(4)$	$\mathbb{C}(8)$	$\mathbb{R}(16)$	$\mathbb{R}(16) \oplus \mathbb{R}(16)$	$\mathbb{R}(32)$	$\mathbb{C}(32)$	$\mathbb{H}(32)$
$\mathbb{H}(2)$	$\mathbb{C}(4)$	$\mathbb{R}(8)$	$\mathbb{R}(8) \oplus \mathbb{R}(8)$	$\mathbb{R}(16)$	$\mathbb{C}(16)$	$\mathbb{H}(16)$	$\mathbb{H}(16) \oplus \mathbb{H}(16)$
$\mathbb{C}(2)$	$\mathbb{R}(4)$	$\mathbb{R}(4) \oplus \mathbb{R}(4)$	$\mathbb{R}(8)$	$\mathbb{C}(8)$	$\mathbb{H}(8)$	$\mathbb{H}(8) \oplus \mathbb{H}(8)$	$\mathbb{H}(16)$
$\mathbb{R}(2)$	$\mathbb{R}(2) \oplus \mathbb{R}(2)$	$\mathbb{R}(4)$	$\mathbb{C}(4)$	$\mathbb{H}(4)$	$\mathbb{H}(4) \oplus \mathbb{H}(4)$	$\mathbb{H}(8)$	$\mathbb{C}(16)$
$\mathbb{R} \oplus \mathbb{R}$	$\mathbb{R}(2)$	$\mathbb{C}(2)$	$\mathbb{H}(2)$	$\mathbb{H}(2) \oplus \mathbb{H}(2)$	$\mathbb{H}(4)$	$\mathbb{C}(8)$	$\mathbb{R}(16)$
$\mathbb{R}$	$\mathbb{C}$	$\mathbb{H}$	$\mathbb{H} \oplus \mathbb{H}$	$\mathbb{H}(2)$	$\mathbb{C}(4)$	$\mathbb{R}(8)$	$\mathbb{R}(8) \oplus \mathbb{R}(8)$

$\xrightarrow[s]{}$

As nice as this is, it might seem that in order to classify real Clifford algebras in arbitrary (albeit finite) dimension we have to do lots of work. Luckily this is not the case, which explains *a posteriori* why I have restricted myself to an 8×8 corner, the so-called Clifford chessboard. It turns out that the real Clifford algebras have simpler periodicities, which are an easy consequence of Theorem 2.3. We call them the **Bott periodicities**.

Corollary 2.6. *For all $n, s, t \geq 0$, the following are isomorphisms of real algebras:*

1. $C\ell(n+8,0) \cong C\ell(n,0) \otimes_{\mathbb{R}} \mathbb{R}(16)$,
2. $C\ell(0,n+8) \cong C\ell(0,n) \otimes_{\mathbb{R}} \mathbb{R}(16)$, *and*
3. $C\ell(s+4,t+4) \cong C\ell(s,t) \otimes_{\mathbb{R}} \mathbb{R}(16)$.

Proof. This follows directly from repeated application of Theorem 2.3 and the following isomorphisms:

$$C\ell(1,1)^{\otimes 4} \cong \mathbb{R}(16) \qquad \text{and} \qquad C\ell(2,0)^{\otimes 2} \otimes C\ell(0,2)^{\otimes 2} \cong \mathbb{R}(16) \ .$$

$\square$

Theorem 2.7 (Classification theorem)**.** *The Clifford algebra $C\ell(s,t)$ is isomorphic to the real associative algebras in the following table, where $d = s + t$:*

$s - t \mod 8$	$C\ell(s,t)$
0, 6	$\mathbb{R}\left(2^{d/2}\right)$
7	$\mathbb{R}\left(2^{(d-1)/2}\right) \oplus \mathbb{R}\left(2^{(d-1)/2}\right)$
1, 5	$\mathbb{C}\left(2^{(d-1)/2}\right)$
2, 4	$\mathbb{H}\left(2^{(d-2)/2}\right)$
3	$\mathbb{H}\left(2^{(d-3)/2}\right) \oplus \mathbb{H}\left(2^{(d-3)/2}\right)$

The proof follows from the Clifford chessboard and the periodicities and it is simply a matter of bookkeeping.

2.3.1 The even subalgebra of the Clifford algebra

In the study of spinor representations it is important to identify the even subalgebra $C\ell(s,t)_0$ of the Clifford algebras $C\ell(s,t)$ as an *ungraded* real associative algebra. Recall that $C\ell(s,t)_0$ is the fixed subalgebra under the automorphism induced by the orthogonal transformation in $O(s,t)$ which sends $x \mapsto -x$ for all $x \in \mathbb{R}^{s,t}$. This means that every element in $C\ell(s,t)_0$ can be written as a linear combination of products of an even number of elements in the image in $C\ell(s,t)$ of $\mathbb{R}^{s,t}$.

Luckily, $C\ell(s,t)_0$ can be determined from the Clifford algebra one dimension lower.

Proposition 2.8. *For all $s,t \geq 0$, we have the following isomorphisms of ungraded real associative algebras:*

$$C\ell(s,t) \cong C\ell(s+1,t)_0 \cong C\ell(t,s+1)_0 \,.$$

In particular, we have that $C\ell(s,t)_0 \cong C\ell(t,s)_0$.

Proof. We will prove one of the isomorphisms and leave the other as an exercise. Let us define $\phi : \mathbb{R}^{s,t} \to C\ell(s+1,t)_0$ by $\phi(x) = xe_{s+1}$, where we write $\mathbb{R}^{s+1,t} = \mathbb{R}^{s,t} \oplus \mathbb{R}e_{s+1}$. We check that ϕ is a Clifford map:

$$\phi(x)^2 = xe_{s+1}xe_{s+1} = -x^2e_{s+1}^2 = x^2 = -Q(x)\mathbf{1} \,,$$

whence it extends uniquely to an algebra homomorphism $\Phi : C\ell(s,t) \to C\ell(s+1,t)_0$. It is clearly injective on generators and counting dimension, we conclude Φ is an isomorphism. □

As a corollary of the classification theorem 2.7, we immediately have a classification of the $C\ell(s,t)_0$.

Corollary 2.9. *The even Clifford algebra $C\ell(s,t)_0$ is isomorphic to the real associative algebras in the following table, where $d = s+t$:*

$s-t \mod 8$	$C\ell(s,t)_0$
1,7	$\mathbb{R}\left(2^{(d-1)/2}\right)$
0	$\mathbb{R}\left(2^{(d-2)/2}\right) \oplus \mathbb{R}\left(2^{(d-2)/2}\right)$
2,6	$\mathbb{C}\left(2^{(d-2)/2}\right)$
3,5	$\mathbb{H}\left(2^{(d-3)/2}\right)$
4	$\mathbb{H}\left(2^{(d-4)/2}\right) \oplus \mathbb{H}\left(2^{(d-4)/2}\right)$

2.4 Classification of complex Clifford algebras

Having determined the real Clifford algebras, it is a simple matter to use Proposition 2.2 and determine the complex Clifford algebras. It is however easier to derive the complex Bott periodicity directly.

Proposition 2.10. *For all $n \geq 0$ there is an isomorphism of complex associative algebras*

$$\mathbb{C}\ell(n+2) \cong \mathbb{C}\ell(n) \otimes_{\mathbb{C}} \mathbb{C}(2) \,.$$

Proof. Write $\mathbb{C}^{n+2} = \mathbb{C}^n \oplus \mathbb{C}e_1 \oplus \mathbb{C}e_2$ and define a complex linear map

$$\phi : \mathbb{C}^{n+2} \longrightarrow \mathbb{C}\ell(n) \otimes_{\mathbb{C}} \mathbb{C}(2)$$

by

$$\phi(x) = x \otimes \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix} \qquad \phi(e_1) = \mathbf{1} \otimes \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix} \qquad \text{and} \qquad \phi(e_2) = \mathbf{1} \otimes \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}$$

for all $x \in \mathbb{C}^n$. One checks that ϕ is Clifford and that the induced map $\Phi : \mathbb{C}\ell(n+2) \to \mathbb{C}\ell(n) \otimes_{\mathbb{C}} \mathbb{C}(2)$, being injective on generators and mapping between equidimensional spaces, is an isomorphism. □

The classification of complex Clifford algebras is then an easy corollary.

Corollary 2.11. *For every $n \geq 0$, the complex Clifford algebra $\mathbb{C}\ell(n)$ is isomorphic to*

$$\mathbb{C}\ell(n) \cong \begin{cases} \mathbb{C}\left(2^{n/2}\right) & \text{if } n \text{ is even,} \\ \mathbb{C}\left(2^{(n-1)/2}\right) \oplus \mathbb{C}\left(2^{(n-1)/2}\right) & \text{if } n \text{ is odd.} \end{cases}$$

Proof. This follows easily from complex Bott periodicity and the "initial conditions" $\mathbb{C}\ell(0) \cong \mathbb{C}$ and $\mathbb{C}\ell(1) \cong \mathbb{C} \oplus \mathbb{C}$. $\square$