Lecture 3: Spinor representations

Yes now I've met me another spinor...
— Suzanne Vega (with apologies)

It was Élie Cartan, in his study of representations of simple Lie algebras, who came across representations of the orthogonal Lie algebra which were not tensorial; that is, not contained in any tensor product of the fundamental (vector) representation. These are the so-called spinorial representations. His description [Car38] of the spinorial representations was quite complicated ("fantastic" according to Dieudonné's review of Chevalley's book below) and it was Brauer and Weyl [BW35] who in 1935 described these representations in terms of Clifford algebras. This point of view was further explored in Chevalley's book [Che54] which is close to the modern treatment. This lecture is devoted to the Pin and Spin groups and to a discussion of their (s)pinorial representations.

3.1 The orthogonal group and its Lie algebra

Throughout this lecture we will let (V,Q) be a real finite-dimensional quadratic vector space with Q nondegenerate. We will drop explicit mention of Q, whence the Clifford algebra shall be denoted $\mathrm{C}\ell(\mathrm{V})$ and similarly for other objects which depend on Q. We will let B denote the bilinear form defining Q.

We start by defining the group $\mathrm{O}(\mathrm{V})$ of orthogonal transformations of V:

$$\mathrm{O}(\mathrm{V}) = \{a : \mathrm{V} \to \mathrm{V} | \mathrm{Q}(av) = \mathrm{Q}(v) \quad \forall v \in \mathrm{V}\} \ . \tag{51}$$

We write $\mathrm{O}(\mathbb{R}^{s,t}) = \mathrm{O}(s,t)$ and $\mathrm{O}(n)$ for $\mathrm{O}(n,0)$. If $a \in \mathrm{O}(\mathrm{V})$, then $\det a = \pm 1$. Those $a \in \mathrm{O}(\mathrm{V})$ with $\det a = 1$ define the special orthogonal group $\mathrm{SO}(\mathrm{V})$. If V is either positive- or negative-definite then $\mathrm{SO}(\mathrm{V})$ is connected: otherwise it has two connected components. This can be inferred by the fact that the connectedness of a Lie group is controlled by that of its maximal compact subgroup, which in the case of $\mathrm{SO}(s,t)$, for $s,t>0$, is

$$\mathrm{S}(\mathrm{O}(s) \times \mathrm{O}(t)) = \{(a,b) \in \mathrm{O}(s) \times \mathrm{O}(t) | \det a = \det b\} ,$$

which has two connected components. The Lie algebra $\mathfrak{so}(\mathrm{V})$ of $\mathrm{SO}(\mathrm{V})$ is defined by

$$\mathfrak{so}(\mathrm{V}) = \{\mathrm{X} : \mathrm{V} \to \mathrm{V} | \mathrm{B}(\mathrm{X}u, v) = -\mathrm{B}(u, \mathrm{X}v) \quad \forall u, v \in \mathrm{V}\} \ . \tag{52}$$

As a vector space, $\mathfrak{so}(\mathrm{V}) \cong \Lambda^2\mathrm{V}$, where the skewsymmetric endomorphism $u \curlywedge v \in \mathfrak{so}(\mathrm{V})$ corresponding to $u \wedge v \in \Lambda^2\mathrm{V}$ is defined by

$$(u \curlywedge v)(x) = \mathrm{B}(u,x)v - \mathrm{B}(v,x)u \ . \tag{53}$$

It is easy to check that $u \curlywedge v \in \mathfrak{so}(\mathrm{V})$ as it is to compute the commutator

$$[u \curlywedge v, x \curlywedge y] = \mathrm{B}(u,x)v \curlywedge y - \mathrm{B}(u,y)v \curlywedge x - \mathrm{B}(v,x)u \curlywedge y + \mathrm{B}(v,y)u \curlywedge x \ . \tag{54}$$

The Clifford algebra $\mathrm{C}\ell(\mathrm{V})$ being associative, becomes a Lie algebra under the commutator and contains $\mathfrak{so}(\mathrm{V})$ as a Lie subalgebra via the embedding

$$\rho : \mathfrak{so}(\mathrm{V}) \to \mathrm{C}\ell(\mathrm{V}) \qquad \text{where} \qquad \rho(u \curlywedge v) = \tfrac{1}{4}(uv - vu) \ . \tag{55}$$

Indeed, it is a simple calculation using the Clifford relation $uv = -vu - 2\mathrm{B}(u,v)\mathbf{1}$ to show that

$$[\rho(u \curlywedge v), x] = \mathrm{B}(u,x)v - \mathrm{B}(v,x)u = (u \curlywedge v)(x) \ , \tag{56}$$

and hence that

$$[\rho(u \curlywedge v), \rho(x \curlywedge y)] = \mathrm{B}(u,x)\rho(v \curlywedge y) - \mathrm{B}(u,y)\rho(v \curlywedge x) - \mathrm{B}(v,x)\rho(u \curlywedge y) + \mathrm{B}(v,y)\rho(u \curlywedge x) \ , \tag{57}$$

whence ρ is an injective Lie algebra homomorphism.

Exponentiating $\mathfrak{so}(V)$ in $\mathrm{End}(V)$ generates the identity component $SO_0(V)$ of $SO(V)$, whereas exponentiating $\rho(\mathfrak{so}(V))$ in $C\ell(V)$ generates a covering group of $SO_0(V)$. We will see this in full generality below, but let us motivate this with an example. Suppose that V contains a positive-definite plane with orthonormal basis e_1, e_2. Then relative to this basis, the restriction to this plane of $e_1 \curlywedge e_2 \in \mathfrak{so}(V)$ has matrix

$$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \tag{58}$$

whose exponential is

$$a(\theta) = \exp(\theta(e_1 \curlywedge e_2)) = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}, \tag{59}$$

whence, in particular, $a(2\pi)$ is the identity matrix. On the other hand, exponentiating the image of the same Lie algebra element $\rho(e_1 \curlywedge e_2) = \frac{1}{2} e_1 e_2$ in $C\ell(V)$ we obtain

$$b(\theta) = \exp(\tfrac{1}{2}\theta e_1 e_2) = \cos(\tfrac{1}{2}\theta)\mathbf{1} + \sin(\tfrac{1}{2}\theta) e_1 e_2, \tag{60}$$

using that $(e_1 e_2)^2 = -\mathbf{1}$. In particular we see that $b(2\pi) = -\mathbf{1}$, so that the periodicity of $b(\theta)$ is 4π. In other words, it suggests that the Lie group generated by exponentiating $\mathfrak{so}(V)$ in $C\ell(V)$ is a double cover of $SO_0(V)$. We will see that this is indeed the case.

3.2 Pin and Spin

Definition 3.1. The **Pin group** $\mathrm{Pin}(V)$ of (V,Q) is the subgroup of (the group of units of) $C\ell(V)$ generated by $v \in V$ with $Q(v) = \pm 1$. In other words, every element of $\mathrm{Pin}(V)$ is of the form $u_1 \cdots u_r$ where $u_i \in V$ and $Q(u_i) = \pm 1$. We will write $\mathrm{Pin}(s,t)$ for $\mathrm{Pin}(\mathbb{R}^{s,t})$ and $\mathrm{Pin}(n)$ for $\mathrm{Pin}(n,0)$.

Let $v \in V \subset C\ell(V)$ and let $Q(v) \neq 0$. Then v is invertible in $C\ell(V)$ and $v^{-1} = -v/Q(v)$. We define, by analogy with the case of a Lie group, the **adjoint action** $\mathrm{Ad}_v : V \to V$, by

$$\mathrm{Ad}_v(x) = v x v^{-1} = \frac{-1}{Q(v)} v x v = \frac{-1}{Q(v)}(-xv - 2B(x,v)\mathbf{1})\, v = -x + 2\frac{B(x,v)}{Q(v)} v = -R_v x\,, \tag{61}$$

where R_v stands for the reflection on the hyperplane perpendicular to v and $x \in V$. We can extend this to a group homomorphism from the Pin group: $\mathrm{Ad}_{v_1 \cdots v_p} = \mathrm{Ad}_{v_1} \circ \cdots \circ \mathrm{Ad}_{v_p}$. Since we would prefer not to see the sign on the right-hand side of $\mathrm{Ad}_v(x)$, we define the **twisted adjoint action** by $\widetilde{\mathrm{Ad}}_v(x) = (-v)xv^{-1} = R_v x$ or more generally $\widetilde{\mathrm{Ad}}_a = \tilde{a} x a^{-1}$ for a an element of the Pin group and $a \mapsto \tilde{a}$ the grading automorphism of $C\ell(V)$, which is induced by the orthogonal transformation $v \mapsto -v$. Let $a = u_1 \cdots u_r \in \mathrm{Pin}(V)$, then $\widetilde{\mathrm{Ad}}_a = R_{u_1} \circ \cdots \circ R_{u_r}$. Since reflections are orthogonal transformations, $\widetilde{\mathrm{Ad}}$ defines a group homomorphism $\widetilde{\mathrm{Ad}} : \mathrm{Pin}(V) \to O(V)$. It follows from the following classic result that $\widetilde{\mathrm{Ad}}$ is surjective.

Theorem 3.2 (Cartan–Dieudonné)**.** *Every $g \in O(V)$ is the product of a finite number of reflections $g = R_{u_1} \circ \cdots \circ R_{u_r}$ along non-null lines ($Q(u_i) \neq 0$) and moreover $r \leq \dim V$.*

We will now determine the kernel of $\widetilde{\mathrm{Ad}}$. Let $a \in \mathrm{Pin}(V)$ be in the kernel of $\widetilde{\mathrm{Ad}}$. This means that $\tilde{a} v = v a$ for all $v \in V$. Let us break up $a = a_0 + a_1$ with $a_0 \in C\ell(V)_0$ and $a_1 \in C\ell(V)_1$, whence $\tilde{a} = a_0 - a_1$. Therefore $a \in \ker \widetilde{\mathrm{Ad}}$ if and only if the following pair of equations are satisfied for all $v \in V$:

$$a_0 v = v a_0 \qquad \text{and} \qquad a_1 v = -v a_1\,. \tag{62}$$

Suppose that $v \in V$ with $Q(v) \neq 0$ and consider $a_0 = \alpha + v\beta$, where α and β do not involve v. Since $\alpha \in C\ell(V)_0$ and does not involve v, then $v\alpha = \alpha v$, whereas since $\beta \in C\ell(V)_1$ and does not involve v, then $v\beta = -\beta v$. The first equation in (62) says that $\beta = 0$, whence $a_0 = \alpha$ does not involve v. Repeating this argument for all the elements of an orthonormal basis (e_i) for V, we see that a_0 does not involve any of the e_i and hence must be a multiple of the identity: $a_0 = \alpha\mathbf{1}$ for some $\alpha \in \mathbb{R}$. Similarly, write $a_1 = \gamma + v\delta$, where γ, δ do not involve v. Now we have that $\gamma v = -v\gamma$, whereas $\delta v = v\delta$. The second equation in (62) says that $\delta = 0$, whence $a_1 = \gamma$ does not involve v. Repeating this argument for the basis (e_i), we see that a_1 does not involve any of the e_i and hence must be a multiple of the identity, but $a_1 \in C\ell(V)_1$ whereas $\mathbf{1} \in C\ell(V)_0$, whence $a_1 = 0$. Hence all elements of Pin(V) in the kernel of $\widetilde{\mathrm{Ad}}$ are multiples of the identity. Now let $u_1 \cdots u_p = \alpha\mathbf{1}$ for $Q(u_i) = \pm 1$. Let us compute the norm of this element using the Clifford inner product (41), to arrive at

$$(\alpha\mathbf{1}, \alpha\mathbf{1}) = \alpha^2(\mathbf{1},\mathbf{1}) = (u_1 \cdots u_p, u_1 \cdots u_p) = (\mathbf{1}, (-u_p)\dots(-u_1)u_1 \cdot u_p) = Q(u_1)\cdots Q(u_p)(\mathbf{1},\mathbf{1})\ . \tag{63}$$

Since $(\mathbf{1},\mathbf{1}) \neq 0$ and $Q(u_i) = \pm 1$, it follows that $\alpha^2 = \pm 1$. Since $\alpha \in \mathbb{R}$ the only solutions to this equation are $\alpha = \pm 1$ and hence $\ker \widetilde{\mathrm{Ad}} = \{\pm\mathbf{1}\}$. In summary we have proved

Proposition 3.3. *The following sequence is exact:*

$$1 \longrightarrow \{\pm\mathbf{1}\} \longrightarrow \mathrm{Pin}(V) \xrightarrow{\widetilde{\mathrm{Ad}}} O(V) \longrightarrow 1\ .$$

Exact sequences

A sequence of groups and group homomorphisms

$$1 \longrightarrow A \xrightarrow{i} B \xrightarrow{p} C \longrightarrow 1$$

is said to be *exact* if the kernel of each homomorphism is the image of the preceding one. In the above diagram, 1 denotes the one-element group. This is both an initial and final object in the category of groups, since there is only one homomorphism into it (sending all elements to the identity) and only one homomorphism out of it (sending the identity to the identity). This explains why we have not given names to the homomorphisms $1 \to A$ and $C \to 1$. Exactness at A means that $i : A \to B$ is injective, since its kernel is the image of $1 \to A$, whence consists only of the identity. Similarly, exactness at C says that $p : B \to C$ is surjective, since the kernel of $C \to 1$ is all of C, and that is precisely the image of p. Finally, exactness at B says that the kernel of $p : B \to C$ is precisely the image of $i : A \to B$. Such an exact sequence says that B is an *extension* of C by A.

Finally, let us define the spin group.

Definition 3.4. The **spin group** of (V, Q) is the intersection

$$\mathrm{Spin}(V) = \mathrm{Pin}(V) \cap C\ell(V)_0\ .$$

Equivalently, it consists of elements $u_1 \cdots u_{2p}$, where $u_i \in V$ and $Q(u_i) = \pm 1$. We will write $\mathrm{Spin}(s,t)$ for $\mathrm{Spin}(\mathbb{R}^{s,t})$ and $\mathrm{Spin}(n)$ for $\mathrm{Spin}(n,0)$.

Since for a reflection $R_u \in O(V)$, we have that $\det R_u = -1$, it follows that $\det \widetilde{\mathrm{Ad}}_a = 1$ for $a \in \mathrm{Pin}(V)$ if and only if $a \in \mathrm{Spin}(V)$. Since the kernel of $\widetilde{\mathrm{Ad}}$ belongs to Spin(V), we immediately have the following

Proposition 3.5. *The following sequence is exact:*

$$1 \longrightarrow \{\pm\mathbf{1}\} \longrightarrow \mathrm{Spin}(V) \xrightarrow{\widetilde{\mathrm{Ad}}} SO(V) \longrightarrow 1\ .$$

For V of signature (s,t) with at least one of $s, t \geq 2$, the map $\widetilde{\mathrm{Ad}} : \mathrm{Spin}(V) \to SO(V)$ is a nontrivial covering. This can be shown by exhibiting a continuous path between $\mathbf{1}$ and $-\mathbf{1}$ in Spin(V). Let e_1, e_2 be an orthonormal basis for a positive- or negative-definite plane. That such a plane exists is a consequence of our assumption on the signature of V. Then consider the following continuous (in fact, analytic) curve in Spin(V):

$$a(t) = (e_1 \cos t + e_2 \sin t)(e_2 \sin t - e_1 \cos t) = Q(e_1)\cos(2t)\mathbf{1} + \sin(2t) e_1 e_2\ .$$

We see that $a(0) = \mathrm{Q}(e_1)\mathbf{1}$, whereas $a(\pi/2) = -\mathrm{Q}(e_1)\mathbf{1}$, whence it joins $\mathbf{1}$ to $-\mathbf{1}$.

Finally let us remark that for V either positive- or negative-definite, SO(V) and hence Spin(V) is connected, whereas for indefinite V, Spin(V) has two connected components. Let $\mathrm{Spin}_0(\mathrm{V})$ denote the identity component. In definite or lorentzian signatures, $\mathrm{Spin}_0(\mathrm{V}) \to \mathrm{SO}_0(\mathrm{V})$ is a universal covering, but $\mathrm{Spin}_0(s,t)$ is not simply connected when both $s,t \geq 2$. The simplest interesting examples of spin covers are $\mathrm{SU}(2) \to \mathrm{SO}(3)$ and $\mathrm{SL}(2,\mathbb{C}) \to \mathrm{SO}_0(3,1)$.

3.3 Pinors and spinors

Informally, pinors (resp. spinors) are vectors in an irreducible representation of a Clifford algebra (resp. its even subalgebra) and, by restriction, of the corresponding Pin (resp. Spin) group. In order to define them properly we need to introduce some notation.

Definition 3.6. Let A be a real associative algebra and let $\mathbb{K} = \mathbb{R}$, $\mathbb{C}$ or $\mathbb{H}$. By a $\mathbb{K}$-**representation** of A we mean an $\mathbb{R}$-linear homomorphism

$$\rho : \mathrm{A} \to \mathrm{End}_{\mathbb{K}}(\mathrm{E})$$

for some $\mathbb{K}$-vector space E. Two $\mathbb{K}$-representations $\rho : \mathrm{A} \to \mathrm{End}_{\mathbb{K}}(\mathrm{E})$ and $\rho' : \mathrm{A} \to \mathrm{End}_{\mathbb{K}}(\mathrm{E}')$ are **equivalent** if there is a $\mathbb{K}$-linear isomorphism $f : \mathrm{E} \to \mathrm{E}'$ such that the triangle

$$\begin{tikzcd} & \mathrm{A} \arrow[dl, "\rho"'] \arrow[dr, "\rho'"] & \\ \mathrm{End}_{\mathbb{K}}(\mathrm{E}) \arrow[rr, "\mathrm{Ad}\, f"] & & \mathrm{End}_{\mathbb{K}}(\mathrm{E}') \end{tikzcd}$$

where $\mathrm{Ad}\, f : \mathrm{End}_{\mathbb{K}}(\mathrm{E}) \to \mathrm{End}_{\mathbb{K}}(\mathrm{E}')$ is defined by $\varphi \mapsto f \circ \varphi \circ f^{-1}$. In other words, for all $a \in \mathrm{A}$, we have that $f \circ \rho(a) = \rho'(a) \circ f$.

Quaternionic vector spaces

Because $\mathbb{H}$ is not commutative, one must distinguish between left and right quaternionic vector spaces. This is largely a matter of convention, since quaternionic conjugation relates left and right vector spaces. Throughout these lectures we shall adopt the convention that $\mathbb{H}^n$ is a right quaternionic vector space. In this way, the matrix algebra $\mathbb{H}(n)$ can act $\mathbb{H}$-linearly on $\mathbb{H}^n$ from the left. (The fact that left and right multiplication commute is precisely associativity.) This defines an isomorphism of *real* algebras $\mathbb{H}(n) \cong \mathrm{End}_{\mathbb{H}}(\mathbb{H}^n)$. In fact, notice that $\mathrm{End}_{\mathbb{H}}(\mathrm{E})$ for a quaternionic vector space E is *only* a real algebra! This is because of the nonexistence of $\mathbb{H}$-bilinears.

In Section 1.4.4 we have already seen one example of an $\mathbb{R}$-representation of $\mathrm{C}\ell(\mathrm{V})$, namely $\Lambda\mathrm{V}$. This representation is not irreducible, however.

Definition 3.7. A **pinor representation** of Pin(V) is the restriction of an irreducible representation of $\mathrm{C}\ell(\mathrm{V})$. Similarly, a **spinor representation** of Spin(V) is the restriction of an irreducible representation of $\mathrm{C}\ell(\mathrm{V})_0$. (It is not hard to see that both pinor and spinor representations are irreducible.)

The irreducible representations of $\mathrm{C}\ell(\mathrm{V})$ are easy to determine from the classification of real Clifford algebras in the second lecture. Recall that as a real algebra, $\mathrm{C}\ell(s,t)$ is isomorphic to either $\mathbb{K}(2^n)$ or $\mathbb{K}(2^n) \oplus \mathbb{K}(2^n)$ depending on the signature. The following result can be extracted from [Lan84, § XVII].

Theorem 3.8.

1. *Every irreducible $\mathbb{R}$-representation of the real algebra $\mathbb{R}(n)$ is isomorphic to $\mathbb{R}^n$, where the matrix $\mathrm{A} \in \mathbb{R}(n)$ acts via left matrix multiplication.*

2. *Every irreducible $\mathbb{H}$-representation of the real algebra $\mathbb{H}(n)$ is isomorphic to $\mathbb{H}^n$ as a right $\mathbb{H}$-vector space and where $\mathrm{A} \in \mathbb{H}(n)$ acts via left matrix multiplication.*

3. *Every irreducible $\mathbb{C}$-representation of the real algebra $\mathbb{C}(n)$ is isomorphic either to $\mathbb{C}^n$ with the natural action given by left matrix multiplication by $A \in \mathbb{C}(n)$ or to $\mathbb{C}^n$ with the complex conjugate action given by left matrix multiplication by $\overline{A} \in \mathbb{C}(n)$.*

This result together with the classification of real Clifford algebras, allows us to determine the pinor representations easily. First of all, we notice that because of the third isomorphism in (49), the type of the Clifford algebra does not change, only the dimension does, when we moved diagonally in the table. This means that the type of the representation of $C\ell(s,t)$ only depends on $s-t$ and, moreover, because of Bott periodicity, only on $s-t \pmod 8$. Thus we need only remember one small part of the Clifford chessboard to determine the rest:

$\mathbb{C}(2)$				
$\mathbb{R}(2)$				
$\mathbb{R}\oplus\mathbb{R}$				
$\mathbb{R}$	$\mathbb{C}$	$\mathbb{H}$	$\mathbb{H}\oplus\mathbb{H}$	$\mathbb{H}(2)$

$\xrightarrow{\quad s \quad}$

Notice that if we colour the squares of the chessboard according to whether $C\ell(s,t)$ has one or two inequivalent irreducible representations, then we do indeed end up with a chessboard pattern.

2	1	2	1	2	
1	2	1	2	1	
2	1	2	1	2	
1	2	1	2	1	

$\xrightarrow{\quad s \quad}$

This dichotomy can also be explained by means of the volume element of $C\ell(V)$. Given an ordered orthonormal basis $(e_1^+,\dots,e_s^+,e_1^-,\dots,e_t^-)$ for $\mathbb{R}^{s,t}$, with $Q(e_i^\pm) = \pm 1$, there is associated a **volume element** of $C\ell(s,t)$ defined as the Clifford product $\omega = e_1^+ \cdots e_s^+ e_1^- \cdots e_t^-$.

Lemma 3.9. *The volume element $\omega \in C\ell(s,t)$ satisfies the following properties:*

1. $\omega^2 = (-1)^{s+d(d-1)/2}\mathbf{1}$, *where $d = s+t$,*

2. ω *is central if $s+t$ is odd, and*

3. $\omega v = -v\omega$ *for all $v \in V$, if $s+t$ is even.*

It follows from the first part that the sign of ω^2 depends only on $s-t \pmod 4$:

$$\omega^2 = \begin{cases} \mathbf{1}, & s-t = 0,3 \pmod 4 \\ -\mathbf{1}, & s-t = 1,2 \pmod 4 \end{cases}$$

−	−	+	+	
−	+	+	−	
+	+	−	−	
+	−	−	+	

$\xrightarrow{\quad s \quad}$

Suppose that $s+t$ (equivalently, $s-t$) is odd, so that ω is central. Then if $\omega^2 = \mathbf{1}$ there are two pinor representations $P_\pm$, distinguished by the action of ω: $\omega = \pm 1$ on $P_\pm$. If $s-t = 3 \pmod 8$, $P_\pm$ is quaternionic, whereas if $s-t = 7 \pmod 8$, $P_\pm$ is real. If $\omega^2 = -\mathbf{1}$, so that $s-t = 1,5 \pmod 8$, there are two complex pinor representations P and $\overline{P}$, distinguished by the action of ω: $\omega = \pm i$ on P and $\overline{P}$, respectively.

In summary, the type and dimension of the pinor representations follows from the classification theorem 2.7. Similarly, the type and dimension of the spinor representations, being representations of the even subalgebra $C\ell(V)_0$, follow from Corollary 2.11. It remains to understand how the pinor and spinor representations are related, for which we need a brief scholium about representation theory.

Real, complex and quaternionic representations

Let G be a Lie group, such as Pin(V) or Spin(V). If $\mathbb{K} = \mathbb{R}, \mathbb{C}$ or $\mathbb{H}$, we will denote $\mathrm{Rep}_{\mathbb{K}}(G)$ denote the (symmetric, monoidal) category of $\mathbb{K}$-representations of G, whose objects are $\mathbb{K}$-vector spaces E (with the usual caveat about the case $\mathbb{K} = \mathbb{H}$) together with group homomorphisms $\rho : G \to GL_{\mathbb{K}}(E)$ and where a morphism between $\rho : G \to GL_{\mathbb{K}}(E)$ and $\rho' : G \to GL_{\mathbb{K}}(E')$ is a $\mathbb{K}$-linear map $f : E \to E'$ such that for all $g \in G$, $f \circ \rho(g) = \rho'(g) \circ f$. There are a number of functors relating these categories, which commute with the direct sum of representations, which is the categorical coproduct in $\mathrm{Rep}_{\mathbb{K}}(G)$. These functors are neatly summarised in the following (noncommutative!) diagram, borrowed from [Ada69] via [BtD85]:

$$
\begin{tikzcd}
 & \mathrm{Rep}_{\mathbb{C}}(G) & \\
\mathrm{Rep}_{\mathbb{R}}(G) \arrow[ur, "e_{\mathbb{R}}^{\mathbb{C}}"] & & \mathrm{Rep}_{\mathbb{H}}(G) \arrow[ul, "r_{\mathbb{C}}^{\mathbb{H}}"'] \\
 & \mathrm{Rep}_{\mathbb{C}}(G) \arrow[ul, "r_{\mathbb{R}}^{\mathbb{C}}"] \arrow[ur, "e_{\mathbb{C}}^{\mathbb{H}}"'] \arrow[uu, leftrightarrow, "c"] &
\end{tikzcd} \tag{64}
$$

where c takes a complex representation E to its complex conjugate $c(E) = \overline{E}$, $e_{\mathbb{R}}^{\mathbb{C}}$ and $e_{\mathbb{C}}^{\mathbb{H}}$ are *extension of scalars*, taking a real representation E to its complexification $e_{\mathbb{R}}^{\mathbb{C}}(E) = E \otimes_{\mathbb{R}} \mathbb{C}$ and a complex representation E to its quaternionification $e_{\mathbb{C}}^{\mathbb{H}}(E) = E \otimes_{\mathbb{C}} \mathbb{H}$, and where $r_{\mathbb{R}}^{\mathbb{C}}$ and $r_{\mathbb{C}}^{\mathbb{H}}$ are *restriction of scalars*, so that we simply view a complex representation E as a real representation $r_{\mathbb{R}}^{\mathbb{C}}(E)$ and a quaternionic representation E as a complex representation $r_{\mathbb{C}}^{\mathbb{H}}(E)$. These functors satisfy a number of identities:

$$
\begin{array}{ccc}
c^2 = 1 & r_{\mathbb{R}}^{\mathbb{C}} e_{\mathbb{R}}^{\mathbb{C}} = 2 & c r_{\mathbb{C}}^{\mathbb{H}} = r_{\mathbb{C}}^{\mathbb{H}} \\
c e_{\mathbb{R}}^{\mathbb{C}} = e_{\mathbb{R}}^{\mathbb{C}} & e_{\mathbb{R}}^{\mathbb{C}} r_{\mathbb{R}}^{\mathbb{C}} = 1 + c & e_{\mathbb{C}}^{\mathbb{H}} r_{\mathbb{C}}^{\mathbb{H}} = 2 \\
e_{\mathbb{C}}^{\mathbb{H}} c = e_{\mathbb{C}}^{\mathbb{H}} & r_{\mathbb{R}}^{\mathbb{C}} c = r_{\mathbb{R}}^{\mathbb{C}} & r_{\mathbb{C}}^{\mathbb{H}} e_{\mathbb{C}}^{\mathbb{H}} = 1 + c
\end{array} \tag{65}
$$

where $2E = E \oplus E$, $(1+c)E = E \oplus \overline{E}$, etc.

In the following discussion, $d = s + t$ is the (real) dimension of V. We will let P and S, perhaps with decorations, denote pinor and spinor representations, respectively; although in order to compare them we must view them both as representations of Spin(V).

If d is even, then the volume element $\omega \in C\ell(s,t)_0$ and commutes with $C\ell(s,t)_0$, whence its eigenspaces in the pinor representation will correspond to the spinors representations. By contrast, if d is odd, then $\omega \not\in C\ell(s,t)_0$ and hence $C\ell(s,t) = C\ell(s,t)_0 \oplus C\ell(s,t)_0\omega \cong C\ell(s,t) \otimes_{\mathbb{R}} \mathbb{R}[\omega]$. This means that we will be able to induce a pinor representation of $C\ell(s,t)$ from a spinor representation S of $C\ell(s,t)_0$ essentially by tensoring with $\mathbb{R}[\omega]$: $P = C\ell(s,t) \otimes_{C\ell(s,t)_0} S$. If $s - t = 1,5 \pmod 8$ then $\omega^2 = -\mathbf{1}$ so that $\mathbb{R}[\omega] \cong \mathbb{C}$, whereas if $s - t = 3,7 \pmod 8$ then $\omega^2 = \mathbf{1}$ so that $\mathbb{R}[\omega] \cong \mathbb{R} \oplus \mathbb{R}$. We will use these facts freely in what follows.

3.3.1 $s - t = 0 \pmod 8$

Here $P \cong \mathbb{R}^{2^{d/2}}$ and $S_\pm \cong \mathbb{R}^{2^{(d-2)/2}}$ as vector spaces. The volume element obeys $\omega^2 = \mathbf{1}$, whence $S_\pm$ are the eigenspaces of ω with eigenvalues ± 1 and $P = S_+ \oplus S_-$.

3.3.2 $s - t = 1 \pmod 8$

Here $\mathrm{P} \cong \mathbb{C}^{2^{(d-1)/2}}$ and $\mathrm{S} \cong \mathbb{R}^{2^{(d-1)/2}}$ as vector spaces. The Clifford algebra $\mathrm{C}\ell(s,t) \cong \mathrm{C}\ell(s,t)_0 \otimes \mathbb{C}$, whence $\mathrm{P} \cong e^{\mathbb{C}}_{\mathbb{R}}(\mathrm{S})$. It follows that $\overline{\mathrm{P}} \cong \mathrm{P}$ as representations of $\mathrm{Spin}(s,t)$.

3.3.3 $s - t = 2 \pmod 8$

Here $\mathrm{P} \cong \mathbb{H}^{2^{(d-2)/2}}$ and $\mathrm{S}, \overline{\mathrm{S}} \cong \mathbb{C}^{2^{(d-2)/2}}$ as vector spaces. We have that $\mathrm{P} \cong e^{\mathbb{H}}_{\mathbb{C}}(\mathrm{S})$, whence $r^{\mathbb{H}}_{\mathbb{C}}(\mathrm{P}) \cong \mathrm{S} \oplus \overline{\mathrm{S}}$, the eigenspace decomposition under ω, which obeys $\omega^2 = -\mathbf{1}$.

3.3.4 $s - t = 3 \pmod 8$

Here $\mathrm{P}_\pm \cong \mathbb{H}^{2^{(d-3)/2}}$ and $\mathrm{S} \cong \mathbb{H}^{2^{(d-3)/2}}$ as vector spaces. The Clifford algebra $\mathrm{C}\ell(s,t) \cong \mathrm{C}\ell(s,t)_0 \oplus \mathrm{C}\ell(s,t)_0$ and hence $\mathrm{P}_\pm \cong \mathrm{S}$.

3.3.5 $s - t = 4 \pmod 8$

Here $\mathrm{P} \cong \mathbb{H}^{2^{(d-2)/2}}$ and $\mathrm{S}_\pm \cong \mathbb{H}^{2^{(d-4)/2}}$ as vector spaces. We have that $\mathrm{P} \cong \mathrm{S}_+ \oplus \mathrm{S}_-$ is the eigenspace decomposition of ω, which obeys $\omega^2 = \mathbf{1}$.

3.3.6 $s - t = 5 \pmod 8$

Here $\mathrm{P}, \overline{\mathrm{P}} \cong \mathbb{C}^{2^{(d-1)/2}}$ and $\mathrm{S} \cong \mathbb{H}^{2^{(d-3)/2}}$ as vector spaces. The Clifford algebra $\mathrm{C}\ell(s,t) \cong \mathrm{C}\ell(s,t)_0 \otimes_{\mathbb{R}} \mathbb{C}$ and hence $\mathrm{P} \cong r^{\mathbb{H}}_{\mathbb{C}}(\mathrm{S})$. It follows that $\overline{\mathrm{P}} \cong \mathrm{P}$ as representations of $\mathrm{Spin}(s,t)$.

3.3.7 $s - t = 6 \pmod 8$

Here $\mathrm{P} \cong \mathbb{R}^{2^{d/2}}$ and $\mathrm{S}, \overline{\mathrm{S}} \cong \mathbb{C}^{2^{(d-2)/2}}$. Then $\mathrm{P} \cong r^{\mathbb{C}}_{\mathbb{R}}(\mathrm{S})$, so that $e^{\mathbb{C}}_{\mathbb{R}}(\mathrm{P}) \cong \mathrm{S} \oplus \overline{\mathrm{S}}$ is the eigenspace decomposition of ω, which obeys $\omega^2 = -\mathbf{1}$, acting on the complexification of P.

3.3.8 $s - t = 7 \pmod 8$

Here $\mathrm{P}_\pm \cong \mathbb{R}^{2^{(d-1)/2}}$ and $\mathrm{S} \cong \mathbb{R}^{2^{(d-1)/2}}$ as vector spaces. The Clifford algebra $\mathrm{C}\ell(s,t) \cong \mathrm{C}\ell(s,t)_0 \oplus \mathrm{C}\ell(s,t)_0$ and hence $\mathrm{P}_\pm \cong \mathrm{S}$.

We can summarise and paraphrase these results by saying that in even dimensions the pinor representation (or if $s - t = 6 \pmod 8$, its complexification) decomposes into the direct sum of two equidimensional spinor representations, whereas in odd dimensions, we must distinguish several cases: if $s - t = 3, 7 \pmod 8$ then each of the two pinor representations is isomorphic to the unique spinor representation, whilst if $s - t = 1, 5 \pmod 8$ then the two complex pinor representations are isomorphic either to the complexification of the unique spinor representation, if $s - t = 1 \pmod 8$, or to the restriction of scalars of the unique quaternionic spinor representation, if $s - t = 5 \pmod 8$.

3.4 Inner products for pinors and spinors

The pinor and spinor representations have inner products which are Spin(V) invariant. In fact, the precise statement, which can be found together with a complete discussion of this topic in [Har90], requires us to review the Clifford involutions.

Clifford involutions

There are three natural involutions of the Clifford algebra $C\ell(V)$:

1. the grading automorphism $\alpha \mapsto \tilde{\alpha}$, which extends the orthogonal transformation $v \mapsto -v$ on V, e.g., $\widetilde{u_1 \cdots u_p} = (-1)^p u_1 \cdots u_p$;
2. the **check involution** $\alpha \to \check{\alpha}$, which is the antiautomorphism of $C\ell(V)$ defined by reversing the order of the generators in every monomial, e.g., $(u_1 \ldots u_p)\check{} = u_p \cdots u_1$; and
3. the **hat involution** $\alpha \mapsto \hat{\alpha}$, obtained by combining the previous two.

If $\alpha \in C\ell(V)$ comes from $\Lambda^p V$ under the isomorphism $C\ell(V) \cong \Lambda V$, then $\tilde{\alpha}$, $\check{\alpha}$ and $\hat{\alpha}$ will be $\pm\alpha$ according the following signs:

$p \mod 4$	0	1	2	3
$\tilde{}$	+	−	+	−
$\check{}$	+	−	−	+
$\hat{}$	+	+	−	−

Notice that on $C\ell(V)_0$, the hat and check involutions agree. This is called the **canonical involution** of $C\ell(V)_0$.

The following theorem can be found in [Har90, Chapter 13].

Theorem 3.10. *There exists an inner product $\langle -,-\rangle$ on every spinor representation S such that*

$$\langle ax, y\rangle = \langle x, \hat{a}y\rangle \qquad \text{for all } a \in C\ell(V)_0 \text{ and } x, y \in S. \tag{66}$$

There exist inner products $\hat{\varepsilon}$ and $\check{\varepsilon}$ on the pinor representation P (possibly taking the direct sum of the two irreducible pinor representations when appropriate) such that

$$\check{\varepsilon}(ax, y) = \check{\varepsilon}(x, \check{a}y) \qquad \text{and} \qquad \hat{\varepsilon}(ax, y) = \hat{\varepsilon}(x, \hat{a}y). \tag{67}$$

Moreover all seven types of inner products (real symmetric, real symplectic, complex symmetric, complex symplectic, complex hermitian, quaternionic hermitian and quaternionic skewhermitian) appear!

The spinor representations (with these Spin(V)-invariant inner products) are behind most of the isomorphisms between the following low-dimensional Lie groups:

$$\begin{array}{lll}
\mathrm{Spin}(2) \cong \mathrm{U}(1) & \mathrm{Spin}(6) \cong \mathrm{SU}(4) & \mathrm{Spin}(5,1)_0 \cong \mathrm{SL}(2,\mathbb{H}) \\
\mathrm{Spin}(3) \cong \mathrm{Sp}(1) & \mathrm{Spin}(2,1)_0 \cong \mathrm{SL}(2,\mathbb{R}) & \mathrm{Spin}(2,2)_0 \cong \mathrm{SL}(2,\mathbb{R}) \times \mathrm{SL}(2,\mathbb{R}) \\
\mathrm{Spin}(4) \cong \mathrm{Sp}(1) \times \mathrm{Sp}(1) & \mathrm{Spin}(3,1)_0 \cong \mathrm{SL}(2,\mathbb{C}) & \mathrm{Spin}(3,2)_0 \cong \mathrm{Sp}(4,\mathbb{R}) \\
\mathrm{Spin}(5) \cong \mathrm{Sp}(2) & \mathrm{Spin}(4,1)_0 \cong \mathrm{Sp}(1,1) & \mathrm{Spin}(4,2)_0 \cong \mathrm{SU}(2,2)
\end{array} \tag{68}$$

In particular, notice the sequence $\mathrm{Spin}_0(2,1) \cong \mathrm{SL}(2,\mathbb{R})$, $\mathrm{Spin}_0(3,1) \cong \mathrm{SL}(2,\mathbb{C})$, $\mathrm{Spin}_0(5,1) \cong \mathrm{SL}(2,\mathbb{H})$, which would suggest that $\mathrm{Spin}_0(9,1)$ would be isomorphic to $\mathrm{SL}(2,\mathbb{O})$ if the octonions were associative and such a group could be defined.