Lecture 4: Spin manifolds

Thus, the existence of a spinor structure appears, on physical grounds, to be a reasonable condition to impose on any cosmological model in general relativity.
— Robert Geroch, 1969

In this lecture we will discuss the notion of a spin structure on a finite-dimensional smooth manifold. We start with some basic notions, just in case the intended audience includes people with little background in differential geometry.

4.1 What is a manifold?

We start with a familiar definition from topology.

Definition 4.1. A (topological) n-**dimensional manifold** is a Hausdorff topological space with a countable basis and which is locally homeomorphic to $\mathbb{R}^n$; that is, every point in M has a neighbourhood which is homeomorphic to $\mathbb{R}^n$.

We shall be interested in doing calculus on manifolds, and this requires introducing a differentiable structure. We assume that we know how to do calculus on $\mathbb{R}^n$ and the point of a differentiable structure is to enable us to do calculus on spaces which are locally "like" $\mathbb{R}^n$ in a way that it is as independent as possible on the precise form of the local homeomorphisms. Please note that we will consider only infinitely differentiable (or *smooth*) structures. This is not necessary, but it is certainly sufficient for our purposes.

Definition 4.2. A **smooth structure** on an n-dimensional manifold M is an **atlas** of **coordinate charts** $\{(U_\alpha, \phi_\alpha)\}_{\alpha\in I}$, for I some indexing set, where $\{U_\alpha\}$ is an open cover of M and $\phi_\alpha : U_\alpha \to \mathbb{R}^n$ are homeomorphisms whose **transition functions** on nonempty overlaps $U_{\alpha\beta} := U_\alpha \cap U_\beta$

$$g_{\alpha\beta} = \phi_\alpha \circ \phi_\beta^{-1} : \phi_\beta(U_{\alpha\beta}) \longrightarrow \phi_\alpha(U_{\alpha\beta})$$

are diffeomorphisms between open subsets of $\mathbb{R}^n$; that is, $g_{\alpha\beta}$ are infinitely differentiable with infinitely differentiable inverses. Two smooth structures $\{(U_\alpha, \phi_\alpha)\}$ and $\{(V_\beta, \psi_\beta)\}$ are **equivalent** if their union is also an atlas. A maximal atlas consists of the union of all atlases in one such equivalence class. A topological manifold with a maximal atlas is called a **smooth manifold**.

Remark 4.3. Notice that not every topological manifold admits a smooth structure and that there are topological manifolds admitting more than one inequivalent smooth structures. For example, it is known that $\mathbb{R}^4$ admits an uncountably infinite number of smooth structures, but for us in this course $\mathbb{R}^n$ will always have the standard smooth structure, unless otherwise explicitly stated.

Let M be a smooth manifold. A function $f : M \to \mathbb{R}$ is **smooth** if for each $\alpha \in I$, $f \circ \phi_\alpha^{-1} : \mathbb{R}^n \to \mathbb{R}$ is smooth as a function of n real variables. Similarly a function $f : M \to \mathbb{R}^p$ is **smooth** if each component function $f_i : M \to \mathbb{R}$, for $i = 1, \dots, p$, is smooth. A map $f : M^n \to N^p$ between smooth manifolds of dimensions n and p, respectively, is **smooth** if for every $m \in M$, $\psi_\beta \circ f : V_\beta \to \mathbb{R}^p$ is smooth for some (and hence all) coordinate charts (V_β, ψ_β) containing the point $f(m)$. Smooth manifolds form the objects of a category whose morphisms are the smooth maps between them. The isomorphisms in that category are the called **diffeomorphisms**: namely, smooth maps $f : M \to N$ with a smooth inverse.

We could say a lot more about calculus on manifolds, but perhaps this suffices for now.

4.2 Fibre bundles

The definition of a spin structure on a smooth manifold is phrased in the language of fibre bundles and we introduce this language in this section.

Let G be a Lie group and let F be a smooth manifold with a smooth G-action: $G \times F \to F$. We will assume that G acts effectively so that if $(g, f) \mapsto f$ for all $f \in F$, then $g = 1$, the identity. It is often convenient to write the action as a map $\rho : G \to \mathrm{Aut}(F)$, where Aut(F) is the automorphism group of the fibre. In the most general case, Aut(F) = Diff(F) is the group of diffeomorphisms, but we will be working mostly with vector bundles, for which F is a vector space and Aut(F) = GL(F), whence ρ is a representation of G.

Definition 4.4. A **fibre bundle** (with **structure group** G and **typical fibre** F as above) over M is a smooth surjection $\pi : E \to M$ together with a local triviality condition: every $m \in M$ has a neighbourhood U and a diffeomorphism $\phi_U : \pi^{-1}U \longrightarrow U \times F$ such that the following triangle commutes:

$$\begin{tikzcd} \pi^{-1}U \arrow[rr, "\phi_U"] \arrow[dr, "\pi"'] & & U \times F \arrow[dl, "\mathrm{pr}_1"] \\ & U & \end{tikzcd}$$

and such that on nonempty overlaps $U \cap V$

$$\phi_U \circ \phi_V^{-1}\big|_{\{m\}\times F} = \rho(g_{UV}(m))$$

for some the **transition functions** $g_{UV} : U \cap V \to G$. The manifold M is called the **base** of the fibre bundle, whereas E is called the **total space**. For each $m \in M$, the **fibre** $\pi^{-1}m = \{e \in E | \pi(e) = m\}$ over m is a submanifold of E which is diffeomorphic to F.

The **trivial** bundle with typical fibre F is simply the Cartesian product $M \times F \xrightarrow{\mathrm{pr}_1} M$, in which case we can take the ϕ_U to be the restriction to U of the identity diffeomorphism. Fibre bundles are of course locally trivial, but can be *twisted in the large*. The maps ϕ_U in the general case are called **local trivialisations**. Fibre bundles (with structure group G) over a fixed smooth manifold M are the objects of a category, where a morphism between two fibre bundles $\pi : E \to M$ and $\pi' : E' \to M$ over M is a G-equivariant fibre-preserving smooth map $\varphi : E \to E'$ such that the following triangle commutes:

(69)
$$\begin{tikzcd} E \arrow[rr, "\varphi"] \arrow[dr, "\pi"'] & & E' \arrow[dl, "\pi'"] \\ & M & \end{tikzcd}$$

The restriction of having the same structure group can be lifted and we can equally well consider morphisms between fibre bundles with different structure groups where now the fibre-preserving map $\varphi : E \to E'$ intertwines between the G and G′ actions on the fibres. In any case, if φ is a diffeomorphism, then the two bundles are said to be **equivalent**. A fibre bundle is said to be **trivial** if it is equivalent to the trivial bundle $M \times F$.

A fibre bundle gives rise to some local data from where it can then be reconstructed, up to equivalence. Let $\mathfrak{U} = \{U_\alpha\}_{\alpha\in I}$ be an open cover of M and let $\phi_\alpha : \pi^{-1}U_\alpha \to U_\alpha \times F$ be local trivialisations with transition functions $g_{\alpha\beta} : U_{\alpha\beta} \to G$, defined by $\rho(g_{\alpha\beta}(m)) = \phi_\alpha \circ \phi_\beta^{-1}\big|_{\{m\}\times F}$ as above. Notice that the $\{g_{\alpha\beta}\}$ satisfy the following conditions:

1. $g_{\alpha\alpha}(m) = 1$ for all $m \in U_\alpha$,
2. $g_{\alpha\beta}(m)g_{\beta\alpha}(m) = 1$ for all $m \in U_{\alpha\beta}$, and
3. the **cocycle condition**

(70)
$$g_{\alpha\beta}(m)g_{\beta\gamma}(m)g_{\gamma\alpha}(m) = 1 \qquad \text{for all } m \in U_{\alpha\beta\gamma} := U_\alpha \cap U_\beta \cap U_\gamma.$$

Notice that if we do not demand that α, β and γ be different, then the cocycle condition implies the other two. We will refer to them collectively as the “cocycle conditions”. Notice as well that we are using that G acts effectively on F, otherwise the right-hand sides of these equations would not necessarily be the identity in G, but anything in the kernel of the action ρ. Now the local trivialisations ϕ_α glue to define a diffeomorphism

$$\mathrm{E} \cong \left(\bigsqcup_{\alpha\in\mathrm{I}} \mathrm{U}_\alpha \times \mathrm{F}\right)\Big/ \sim \qquad \text{where } (m,f) \sim (m, \rho(g_{\alpha\beta}(m))f), \text{ for all } m \in \mathrm{U}_{\alpha\beta} \text{ and } f \in \mathrm{F}. \tag{71}$$

From now on we shall drop ρ from the notation and simply write gf for $g \in \mathrm{G}$ acting on $f \in \mathrm{F}$. Notice that the cocycle conditions above are, respectively, the reflexive, symmetry and transitivity conditions for the equivalence relation ∼.

This allows us to construct fibre bundles by gluing local data. Indeed, if we are given an open cover $\mathfrak{U} = \{\mathrm{U}_\alpha\}_{\alpha\in\mathrm{I}}$ for M and functions $g_{\alpha\beta} : \mathrm{U}_{\alpha\beta} \to \mathrm{G}$ on overlaps satisfying the cocycle conditions then we get a fibre bundle by defining E by (71) and the surjection $\pi : \mathrm{E} \to \mathrm{M}$ by the projection $\mathrm{pr}_1 : \mathrm{U}_\alpha \times \mathrm{F} \to \mathrm{U}_\alpha$, which is respected by the equivalence relation. The resulting bundle is a fibre bundle trivialised over $\mathfrak{U}$.

4.2.1 Vector and principal bundles

As mentioned above, a general fibre bundle will have as structure group the diffeomorphism group of the typical fibre, but there are important examples where G is much smaller. For example, if we take F to be a vector space and G to act linearly, then we have a **vector bundle**. Similarly if F ≅ G itself and G acts on G by left multiplication, then we have a **principal** G-**bundle**. In this latter case, there is a well-defined *right* action of G on the total space E of the principal bundle: $(m,g) \mapsto (m, gg')$ which is fibre-preserving and clearly G-equivariant, since associativity of the group multiplication says that left and right multiplications commute. Since right multiplication of G on itself is simply transitive, we have that M ≅ E/G is the quotient of E by this right action of G. One often sees this as the starting point for a definition of a principal bundle.

We can go back and forth between vector and principal bundles via two natural constructions:

$$\begin{tikzcd} \begin{matrix}\text{vector}\\ \text{bundles}\end{matrix} \arrow[r, shift left, "\text{bundle of frames}"] & \begin{matrix}\text{principal}\\ \text{bundles}\end{matrix} \arrow[l, shift left, "\text{associated bundle}"] \end{tikzcd}$$

Given a vector bundle $\mathrm{E} \xrightarrow{\pi} \mathrm{M}$ with typical fibre a vector space F, let us define a principal bundle $\mathrm{GL(E)} \xrightarrow{\Pi} \mathrm{M}$ by declaring the fibre $\mathrm{GL(E)}_m$ to be the set of frames of the vector space E_m. This is a principal homogeneous space (or *torsor*) of the general linear group in that any two frames are related by a unique invertible linear transformation. This means that as a set $\mathrm{GL(E)}_m \cong \mathrm{GL}(\mathrm{E}_m)$, but the isomorphism is not natural: it depends on choosing a reference frame. Nevertheless, in terms of a local trivialisation $\{(\mathrm{U}_\alpha, \phi_\alpha)\}_{\alpha\in\mathrm{I}}$ for E, with transition functions $g_{\alpha\beta} : \mathrm{U}_{\alpha\beta} \to \mathrm{GL(F)}$ we define

$$\mathrm{GL(E)} = \left(\bigsqcup_{\alpha\in\mathrm{I}} \mathrm{U}_\alpha \times \mathrm{GL(F)}\right)\Big/ \sim \qquad \text{where } (m,g) \sim (m, g_{\alpha\beta}(m)g), \text{ for all } m \in \mathrm{U}_{\alpha\beta} \text{ and } g \in \mathrm{GL(F)}.$$

This is then a principal GL(F)-bundle.

Conversely, given a principal G-bundle $\mathrm{P} \xrightarrow{\Pi} \mathrm{M}$ and a finite-dimensional representation $\rho : \mathrm{G} \to \mathrm{GL(F)}$ on a vector space F, we have a right G-action on P × F given by $(p,f)g = (pg, \rho(g^{-1})f)$ for $p \in \mathrm{P}$, $f \in \mathrm{F}$ and $g \in \mathrm{G}$. This action is free because G acts freely on P, and hence the quotient E = (P × F)/G can be given a smooth structure. Since P/G ≅ M, we see that this is a fibre bundle with typical fibre F. The surjection $\pi : \mathrm{E} \to \mathrm{M}$ is induced from the surjection $\Pi : \mathrm{P} \to \mathrm{M}$ which is preserved by the equivalence relation by virtue of $\Pi(pg) = \Pi(p)$. Alternatively, in terms of a local trivialisation $\{\mathrm{U}_\alpha, \phi_\alpha\}_{\alpha\in\mathrm{I}}$ for P with transition functions $g_{\alpha\beta} : \mathrm{U}_{\alpha\beta} \to \mathrm{G}$, we define

$$\mathrm{E} = \left(\bigsqcup_{\alpha\in\mathrm{I}} \mathrm{U}_\alpha \times \mathrm{F}\right)\Big/ \sim \qquad \text{with } (m,f) \sim (m, g_{\alpha\beta}(m)f) \text{ for } m \in \mathrm{U}_{\alpha\beta} \text{ and } f \in \mathrm{F}$$

and $\pi : \mathrm{E} \to \mathrm{M}$ defined by $\pi[(m,f)] = m$.

In summary, the two constructions above relate fibre bundles which are locally trivialisable over the same cover and the corresponding transition functions are simply related. It is largely a matter of choice whether one decides to work with principal bundles and their associated bundles or with vector bundles and their bundles of frames. For the most part we will choose the former.

4.2.2 Equivalence classes of principal bundles

From the above discussion, the emerging picture is one of principal G-bundles defined by data consisting of a trivialising cover $\mathfrak{U} = \{U_\alpha\}_{\alpha\in I}$ and functions $g_{\alpha\beta} : U_{\alpha\beta} \to G$ on double overlaps satisfying the cocycle conditions (70). Different choices of $\mathfrak{U}$ and of cocycles $\{g_{\alpha\beta}\}$ can still give rise to equivalent bundles.

From the definition of the $g_{\alpha\beta}$ in terms of local trivialisations $g_{\alpha\beta} = \phi_\alpha \circ \phi_\beta^{-1}$ one can see that it is still possible to compose ϕ_α with functions $g_\alpha : U_\alpha \to G$ to give rise to a new trivialisation $\phi'_\alpha = g_\alpha \circ \phi_\alpha$ and hence to new transition functions $g'_{\alpha\beta} = g_\alpha \circ g_{\alpha\beta} \circ g_\beta^{-1}$ which still satisfy the cocycle conditions. We say that two cocycles $\{g_{\alpha\beta}\}$ and $\{g'_{\alpha\beta}\}$ are equivalent if $g'_{\alpha\beta} = g_\alpha \circ g_{\alpha\beta} \circ g_\beta^{-1}$ for some "cochain" $\{g_\alpha\}$. We will let $H^1(\mathfrak{U}, G)$ denote the set of equivalence classes of cocycles. It classifies the principal G-bundles trivialised on $\mathfrak{U}$ up to equivalence.

Remark 4.5. Those familiar with sheaf cohomology will recognise $H^1(\mathfrak{U}, G)$ as the first Čech cohomology *set* of the sheaf of germs of smooth functions $M \to G$ relative to the open cover $\mathfrak{U}$. For G nonabelian, this will fail to be a group and be only a pointed set, with distinguished element the isomorphism class of the trivial bundle.

Now suppose that we are given two principal G-bundles defined by local data $(\mathfrak{U} = \{U_\alpha\}_{\alpha\in I}, \{g_{\alpha\beta}\})$ and $(\mathfrak{V} = \{V_\alpha\}_{\alpha\in J}, \{g'_{\alpha\beta}\})$. In order to compare them we would like to define the two bundles relative to the same trivialising cover. This is done by passing to a common *refinement* of $\mathfrak{U}$ and $\mathfrak{V}$. More precisely let $\mathfrak{U} = \{U_\alpha\}_{\alpha\in I}$ be an open cover for M. We say that an open cover $\mathfrak{V} = \{V_\beta\}_{\beta\in J}$ **refines** $\mathfrak{U}$ if there is a reindexing map $j : J \to I$ such that for every $\beta \in J$, $V_\beta \subseteq U_{j(\beta)}$. Now any two open covers $\mathfrak{U} = \{U_\alpha\}_{\alpha\in I}$ and $\mathfrak{V} = \{V_\beta\}_{\beta\in J}$ have a common refinement. For example, we can take $\mathfrak{W} = \{U_\alpha \cap V_\beta\}_{(\alpha,\beta)\in I\times J}$. It is clearly again an open cover and it is clear that it refines both $\mathfrak{U}$ and $\mathfrak{V}$: the reindexing functions $I \times J \to I$ and $I \times J \to J$ are the cartesian projections. (This makes the set of open covers into a **directed set**.) We can then restrict the cocycles $\{g_{\alpha\beta}\}$ defined on $\mathfrak{U}$ and $\{g'_{\alpha\beta}\}$ defined on $\mathfrak{V}$ to $\mathfrak{W}$ and in effect consider them as cocycles on $\mathfrak{W}$ where they can be compared as above. So that two principal bundles defined by local data $(\mathfrak{U} = \{U_\alpha\}_{\alpha\in I}, \{g_{\alpha\beta}\})$ and $(\mathfrak{V} = \{V_\alpha\}_{\alpha\in J}, \{g'_{\alpha\beta}\})$ are equivalent if the restriction of their cocycles to some refinement $\mathfrak{W}$ are equivalent, whence they define the same class in $H^1(\mathfrak{W}, G)$. The way to formalise this is to define define $H^1(M, G) = \varinjlim_{\mathfrak{U}} H^1(\mathfrak{U}, G)$, the direct limit of the restriction maps $H^1(\mathfrak{U}, G) \to H^1(\mathfrak{V}, G)$ for $\mathfrak{V}$ a refinement of $\mathfrak{U}$. Then we see that two principal G-bundles are equivalent if they define the same class in $H^1(M, G)$, which then becomes the set of equivalence classes of principal G-bundles on M. It is a pointed set with the trivial bundle as distinguished element.

4.3 Some vector bundles on riemannian manifolds

We will now specialise to riemannian manifolds.

4.3.1 Orientability and the orthonormal frame bundle

Tangent bundle

Let M be a smooth manifold and $m \in M$ a point. Then by a curve through m we mean a smooth function $t \mapsto c(t)$, with $c(0) = m$. Its velocity at m is the derivative with respect to t evaluated at $t = 0$: $c'(0)$. The space of the velocities at m of all curves though m defines the **tangent space** $T_m M$ of M at m. It is a vector space. The union $TM := \bigcup_{m\in M} T_m M$ can be given the structure of a smooth manifold in such a way that the map $\pi : TM \to M$ which sends $v \in T_m M$ to m is a surjection making it into a vector bundle over M.

A **riemannian manifold** (M, g) is a manifold M together with a **metric** g, which is a smoothly varying family of nondegenerate symmetric bilinear forms on the tangent spaces of M. Notice that we do not demand that g be positive-definite.

Remark 4.6. Although every (paracompact) smooth manifold admits a positive-definite metric, the existence of indefinite metrics often imposes topological restrictions on M. For example, if M is compact (and orientable?) then it admits a lorentzian metric (i.e., one of signature $(1, n-1)$ or $(n-1, 1)$ for $n > 1$) if and only if its Euler characteristic vanishes – a result due to Geroch.

The tangent bundle of a smooth n-dimensional manifold has structure group $\mathrm{GL}(n,\mathbb{R})$, but for a riemannian manifold, the existence of orthonormal frames implies that it is equivalent to a vector bundle with structure group $\mathrm{O}(s,t)$ if the metric has signature (s,t). If $\mathfrak{U} = \{\mathrm{U}_\alpha\}_{\alpha\in\mathrm{I}}$ is a trivialising cover for the tangent bundle, we let $g_{\alpha\beta} : \mathrm{U}_{\alpha\beta} \to \mathrm{O}(s,t)$ be the transition functions for the bundle $\mathrm{O}(\mathrm{M}) \to \mathrm{M}$ of orthonormal frames.

Example 4.7. Let $\mathrm{S}^n \subset \mathbb{R}^{n+1}$ be the unit sphere. For $x \in \mathrm{S}^n$, the tangent space $\mathrm{T}_x\mathrm{S}^n$ is given by those vectors in $\mathbb{R}^{n+1}$ which are perpendicular to x. The orthonormal frame bundle is $\mathrm{O}(n+1) \to \mathrm{S}^n$. Indeed, given $x \in \mathrm{S}^n$ and an orthonormal frame $e_1,\dots,e_n$ for $\mathrm{T}_x\mathrm{S}^n$, the $(n+1)\times(n+1)$-matrix whose first n columns are given by the e_i and whose last column is given by x is orthogonal. Conversely, given $a \in \mathrm{O}(n+1)$, the map $\pi : \mathrm{O}(n+1) \to \mathrm{S}^n$ defined by setting $\pi(a)$ to be the last column of a is such that the fibre at $\pi(a)$ is the set of orthonormal frames for the perpendicular subspace to $\pi(a)$ in $\mathbb{R}^{n+1}$.

A riemannian manifold (M, g) is **oriented** if we can restrict consistently to oriented orthonormal frames or, in other words, whether we can reduce the structure group of TM from $\mathrm{O}(s,t)$ to $\mathrm{SO}(s,t)$. Concretely, this means being able to choose transition functions for the orthonormal frame bundle which lie in $\mathrm{SO}(s,t)$, perhaps relative to a refinement of the trivialising cover. So given $\{g_{\alpha\beta}\}$ taking values in $\mathrm{O}(s,t)$ we ask whether we can find $\{g'_{\alpha\beta}\}$ taking values in $\mathrm{SO}(s,t)$. Let $f_{\alpha\beta}(m) = \det g_{\alpha\beta}(m)$ for $m \in \mathrm{U}_{\alpha\beta}$. Since an orthogonal matrix, independently of the signature, has determinant ± 1, the $f_{\alpha\beta}(m)$ take values in the group $\{\pm 1\}$ of order 2. The cocycle condition for $\{g_{\alpha\beta}\}$ imply the cocycle condition for $\{f_{\alpha\beta}\}$, whence this defines a principal fibre bundle with structure group $\mathbb{Z}_2$. Orientability of M is equivalent to the triviality of this bundle. Indeed, if (and only if) $f_{\alpha\beta}(m) = f_\alpha(m) f_\beta(m)$ for some $\mathbb{Z}_2$-valued "cochain" $f_\alpha : \mathrm{U}_\alpha \to \mathbb{Z}_2$, then we can define $g'_{\alpha\beta}(m) = g_\alpha(m) g_{\alpha\beta}(m) g_\beta^{-1}(m)$ for some $g_\alpha : \mathrm{U}_\alpha \to \mathrm{O}(s,t)$ with $\det g_\alpha(m) = f_\alpha(m)$, whence $g'_{\alpha\beta} : \mathrm{U}_{\alpha\beta} \to \mathrm{SO}(s,t)$ and still satisfies the cocycle conditions. The cohomology class defined by $\{f_{\alpha\beta}\}$ in $\mathrm{H}^1(\mathrm{M},\mathbb{Z}_2)$ which measures the failure of M to being orientable is called the **first Stiefel–Whitney class** of M. Its vanishing is tantamount to orientability. Since $\mathrm{H}^1(\mathrm{M},\mathbb{Z}_2) \cong \mathrm{Hom}(\pi_1(\mathrm{M}),\mathbb{Z}_2)$, if M is simply connected then it is automatically orientable. Even if M is not orientable, there is a double cover, namely the total space of the principal $\mathbb{Z}_2$-bundle defined by the class of $\{f_{\alpha\beta}\}$ in $\mathrm{H}^1(\mathrm{M},\mathbb{Z}_2)$, which is oriented and locally isometric to (M, g).

Remark 4.8. Readers familiar with Čech cohomology will recognise the obstruction of orientability as image of the class in $\mathrm{H}^1(\mathrm{M},\mathrm{O}(s,t))$ corresponding to the orthonormal frame bundle under the last map in the long exact cohomology sequence

$$\mathrm{H}^1(\mathrm{M},\mathrm{SO}(s,t)) \longrightarrow \mathrm{H}^1(\mathrm{M},\mathrm{O}(s,t)) \longrightarrow \mathrm{H}^1(\mathrm{M},\mathbb{Z}_2)$$

coming from the exact sheaf sequence which is induced from the exact sequence of groups

$$1 \longrightarrow \mathrm{SO}(s,t) \longrightarrow \mathrm{O}(s,t) \xrightarrow{\det} \mathbb{Z}_2 \longrightarrow 1$$

Notice that since $\mathrm{O}(s,t)$ and $\mathrm{SO}(s,t)$ are nonabelian groups, there are no $\mathrm{H}^{p>1}$, whence the exact cohomology sequence ends there.

Example 4.9. For $n \geq 2$, the sphere S^n is simply connected, whence it is orientable. In fact, the oriented orthonormal frame bundle is $\mathrm{SO}(n+1) \to \mathrm{S}^n$, with the map given again by the last column of the matrix.

4.3.2 The Clifford bundle and the obstruction to defining a pinor bundle

Any functorial construction on vector spaces — e.g., $\oplus$, $\otimes$, Hom,... — gives rise to a similar construction on vector bundles, and in particular any such construction on representations of G gives rise to similar constructions on associated vector bundles to any principal G-bundle. On a riemannian manifold

(M, g) each tangent space becomes a quadratic vector space, relative to the quadratic form induced from the inner product defined by the metric. Hence one should expect that any functorial construction on quadratic vector spaces should globalise to a similar construction on a riemannian manifold. One such construction is the Clifford algebra, which gives rise to a **Clifford bundle** $C\ell(TM)$. As a vector bundle, $C\ell(TM) \cong \Lambda TM$, but $C\ell(TM)$ is actually a bundle of Clifford algebras. Alternatively we can define it from a local trivialisation of the orthonormal frame bundle $O(M)$:

$$C\ell(TM) = \left(\bigsqcup_{\alpha\in I} U_\alpha \times C\ell(s,t)\right)\Big/\sim \qquad \text{with } (m,c) \sim (m, C\ell(g_{\alpha\beta}(m))c) \text{ for } m \in U_{\alpha\beta} \text{ and } c \in C\ell(s,t),$$

where $C\ell(g_{\alpha\beta}(m))$ is the Clifford algebra automorphism derived functorially from the orthogonal transformation $g_{\alpha\beta}(m)$. Since $C\ell(g_{\alpha\beta}(m))$ are automorphisms of the Clifford algebra, the Clifford product on $C\ell(TM)$ is well-defined.

A natural question, given the existence of the Clifford bundle, is whether there is a vector bundle associated to the pinor representation of the Clifford algebra. If $P(s,t)$ is a pinor representation of $C\ell(s,t)$ one could try to build such a bundle from local data as follows

$$P \stackrel{?}{=} \left(\bigsqcup_{\alpha\in I} U_\alpha \times P(s,t)\right)\Big/\sim \qquad \text{with } (m,p) \sim (m, g_{\alpha\beta}(m)p) \text{ for } m \in U_{\alpha\beta} \text{ and } p \in C\ell(s,t),$$

except that $O(s,t)$ does not act on $P(s,t)$ and hence we don't know what $g_{\alpha\beta}(m)p$ is.

Since $\mathrm{Pin}(s,t)$ does act on $P(s,t)$, we could try to define

$$P \stackrel{?}{=} \left(\bigsqcup_{\alpha\in I} U_\alpha \times P(s,t)\right)\Big/\sim \qquad \text{with } (m,p) \sim (m, \widetilde{g_{\alpha\beta}(m)}p) \text{ for } m \in U_{\alpha\beta} \text{ and } p \in C\ell(s,t),$$

where $\widetilde{g_{\alpha\beta}(m)} \in \mathrm{Pin}(s,t)$ is a lift of $g_{\alpha\beta}(m) \in O(s,t)$. In other words, $\widetilde{\mathrm{Ad}}_{\widetilde{g_{\alpha\beta}(m)}} = g_{\alpha\beta}(m)$, where $\widetilde{\mathrm{Ad}} : \mathrm{Pin}(s,t) \to O(s,t)$ is the surjection in Proposition 3.3. For the above definition to make sense, $\sim$ must be an equivalence relation and this is tantamount to the cocycle condition for $\widetilde{g_{\alpha\beta}(m)} : U_{\alpha\beta} \to \mathrm{Pin}(s,t)$:

$$\widetilde{g_{\alpha\beta}(m)}\,\widetilde{g_{\beta\gamma}(m)}\,\widetilde{g_{\gamma\alpha}(m)} = 1 \qquad \text{for all } m \in U_{\alpha\beta\gamma}.$$

Applying $\widetilde{\mathrm{Ad}}$ to the cocycle conditions, we obtain the cocycle conditions for the $g_{\alpha\beta}(m) : U_{\alpha\beta} \to O(s,t)$, which are satisfied, hence

$$f_{\alpha\beta\gamma}(m) := \widetilde{g_{\alpha\beta}(m)}\,\widetilde{g_{\beta\gamma}(m)}\,\widetilde{g_{\gamma\alpha}(m)} \in \ker \widetilde{\mathrm{Ad}} = \mathbb{Z}_2$$

and hence defines maps $f_{\alpha\beta\gamma} : U_{\alpha\beta\gamma} \to \mathbb{Z}_2$. Moreover $f_{\alpha\beta\gamma}$ is itself a cocycle, in that in quadruple overlaps

$$f_{\alpha\beta\gamma}(m) f_{\alpha\beta\delta}(m) f_{\alpha\gamma\delta}(m) f_{\beta\gamma\delta}(m) = 1 \qquad \text{for all } m \in U_{\alpha\beta\gamma\delta}.$$

Since $\widetilde{\mathrm{Ad}}$ has nontrivial kernel, the lift $\widetilde{g_{\alpha\beta}}(m)$ is not unique and any other lift is related to this by some cochain $f_{\alpha\beta} : U_{\alpha\beta} \to \mathbb{Z}_2$. This changes the cocycle $f_{\alpha\beta\gamma}$ by a coboundary

$$f_{\alpha\beta\gamma}(m) \mapsto f'_{\alpha\beta\gamma}(m) := f_{\alpha\beta\gamma}(m) f_{\alpha\beta}(m) f_{\beta\gamma}(m) f_{\alpha\gamma}(m)\,.$$

In particular, $f'_{\alpha\beta\gamma}$ still satisfies the cocycle condition on quadruple overlaps and its class in $H^2(\mathfrak{U}, \mathbb{Z}_2)$, and hence in $H^2(M, \mathbb{Z}_2)$, is unchanged. If (and only if) this class vanishes, will we be able to lift the $g_{\alpha\beta}$ to $\mathrm{Pin}(s,t)$ in such a way that the cocycle conditions are satisfied. Indeed, the class of $f_{\alpha\beta\gamma}$ in $H^2(M, \mathbb{Z}_2)$ vanishes if on some "good" cover $\mathfrak{U}$, $f_{\alpha\beta\gamma}(m) = f_{\alpha\beta}(m) f_{\beta\gamma}(m) f_{\alpha\gamma}(m)$ for some $f_{\alpha\beta} : U_{\alpha\beta} \to \mathbb{Z}_2$. This being the case, then $g'_{\alpha\beta}(m) = \widetilde{g_{\alpha\beta}(m)} f_{\alpha\beta}(m)$ is our desired $\mathrm{Pin}(s,t)$-valued cocycle. The class in $H^2(M, \mathbb{Z}_2)$ defined by the $f_{\alpha\beta\gamma}$ is essentially the **second Stiefel–Whitney class** of M, and the pinor bundle can be defined if and only if this class vanishes.

We can view the same class appearing in the more traditional approach to defining a spin structure, to which we now turn.

4.3.3 Spin structures

Let (M, g) be an orientable riemannian manifold of signature (s, t) and let $SO(M) \to M$ denote the bundle of oriented orthonormal frames.

Definition 4.10. A **spin structure** on (M, g) is a principal $\mathrm{Spin}(s, t)$-bundle $\mathrm{Spin}(M) \to M$ together with a bundle morphism

$$\begin{tikzcd} \mathrm{Spin}(M) \arrow[rr, "\varphi"] \arrow[dr] & & SO(M) \arrow[dl] \\ & M & \end{tikzcd}$$

which restricts fibrewise to the covering homomorphism $\widetilde{\mathrm{Ad}} : \mathrm{Spin}(s, t) \to SO(s, t)$ of Proposition 3.5.

Spin structures need not exist and even if they do they need not be unique. To understand the obstruction let us try to build a spin bundle starting with a trivialisation $(\mathfrak{U}, \{g_{\alpha\beta}\})$ of $SO(M)$. We choose $\widetilde{g_{\alpha\beta}(m)} \in \mathrm{Spin}(s, t)$ such that under $\widetilde{\mathrm{Ad}} : \mathrm{Spin}(s, t) \to SO(s, t)$, $\widetilde{g_{\alpha\beta}(m)} \mapsto g_{\alpha\beta}(m)$. This choice is not unique, of course: any other choice $\widetilde{g'_{\alpha\beta}(m)}$ is related to $\widetilde{g_{\alpha\beta}(m)}$ by multiplication with some $f_{\alpha\beta}(m) \in \ker \widetilde{\mathrm{Ad}} = \mathbb{Z}_2$: $\widetilde{g'_{\alpha\beta}(m)} = \widetilde{g_{\alpha\beta}(m)} f_{\alpha\beta}(m)$. We would build the spin bundle $\mathrm{Spin}(M)$ as usual by

$$\mathrm{Spin}(M) \stackrel{?}{=} \left(\bigsqcup_{\alpha \in I} U_\alpha \times \mathrm{Spin}(s, t) \right) \Big/ \sim \qquad \text{with } (m, s) \sim (m, \widetilde{g_{\alpha\beta}(m)} s) \text{ for } m \in U_{\alpha\beta} \text{ and } s \in \mathrm{Spin}(s, t),$$

except that, for this to make sense, the $\widetilde{g_{\alpha\beta}(m)}$ should satisfy the cocycle condition. As in the case of the construction of the pinor bundle, the obstruction is the class of $f_{\alpha\beta\gamma}(m) = \widetilde{g_{\alpha\beta}(m)}\widetilde{g_{\beta\gamma}(m)}\widetilde{g_{\gamma\alpha}(m)}$ in $H^2(M, \mathbb{Z}_2)$, which is again the second Stiefel–Whitney class of M. If and only if this class vanishes does (M, g) admit a spin structure. Assuming the class vanishes, then one can ask whether the spin structure is unique. Spin structures are in bijective correspondence with the inequivalent lifts $\widetilde{g_{\alpha\beta}}$ of $g_{\alpha\beta}$. As mentioned above, any two lifts are related by multiplication by $f_{\alpha\beta} : U_{\alpha\beta} \in \mathbb{Z}_2$. The cocycle conditions of the two lifts implies the cocycle condition of $f_{\alpha\beta}$, whence it defines a class in $H^1(M, \mathbb{Z}_2)$. If (and only if) this class is trivial, so that $f_{\alpha\beta}(m) = f_\alpha(m) f_\beta(m)$ for some $f_\alpha : U_\alpha \to \mathbb{Z}_2$, do the two lifts yield equivalent spin bundles. In summary, spin structures are classified by $H^1(M, \mathbb{Z}_2) \cong \mathrm{Hom}(\pi_1(M), \mathbb{Z}_2)$, whence it usually comes down to assigning signs to noncontractible loops consistently.

Remark 4.11. Readers familiar with Čech cohomology will recognise the obstruction of the existence of a spin structure as the image of the class of $SO(M)$ in $H^1(M, SO(s, t))$ under the connecting map in the long exact cohomology sequence

$$H^1(M, \mathbb{Z}_2) \longrightarrow H^1(M, \mathrm{Spin}(s, t)) \longrightarrow H^1(M, SO(s, t)) \longrightarrow H^2(M, \mathbb{Z}_2)$$

coming from the exact sheaf sequence which is induced from the exact sequence of groups

$$1 \longrightarrow \mathbb{Z}_2 \longrightarrow \mathrm{Spin}(s, t) \xrightarrow{\widetilde{\mathrm{Ad}}} SO(s, t) \longrightarrow 1$$

Again notice that since $SO(s, t)$ and $\mathrm{Spin}(s, t)$ are (in general) nonabelian, there are no $H^{p>1}$, whence the exact cohomology sequence ends there. Indeed a principal $SO(s, t)$-bundle admits a Spin lift if and only its image in $H^2(M, \mathbb{Z}_2)$ under the connecting homomorphism vanishes and the "difference" of any two lifts lives in $H^1(M, \mathbb{Z}_2)$.

Example 4.12. For $n \geq 2$, the sphere S^n admits a unique spin structure, and indeed $\mathrm{Spin}(S^n) = \mathrm{Spin}(n+1)$ and the bundle morphism $\mathrm{Spin}(S^n) \to SO(S^n)$ is the covering homomorphism $\mathrm{Spin}(n+1) \to SO(n+1)$.

Example 4.13. The circle S^1 has two inequivalent spin structures which, in some quarters at least, go by the names of *Ramond* and *Neveu–Schwarz*. (This is *not* a joke.)

Example 4.14. A compact Riemann surface Σ of genus g admits 2^{2g} inequivalent spin structures. The second Stiefel–Whitney class vanishes because if it the reduction $\mod 2$ of the Euler class and the Euler characteristic is even $(= 2 - 2g)$. The inequivalent spin structures are classified by homomorphisms $\mathrm{Hom}(\pi_1(\Sigma), \mathbb{Z}_2)$. Now the fundamental group of Σ is generated by $2g$ elements $A_1, \ldots, A_g, B_1, \ldots, B_g$ subject to the relation $[A_1, B_1][A_2, B_2]\cdots[A_g, B_g] = 1$, where $[A, B] = ABA^{-1}B^{-1}$ is the (group-theoretical) commutator of A, B. Every homomorphism is determined by what it does on generators, subject to the relation being satisfied. Clearly, though, since $\mathbb{Z}_2$ is abelian, any homomorphism from the free group generated by the A_i and the B_i automatically preserves the relation. Thus every spin structure is specified by associating a sign to every generator. For the case of genus 1, there are four spin structures which, in some quarters at least, are called *Neveu–Schwarz/Neveu–Schwarz*, *Neveu–Schwarz/Ramond*, *Ramond/Neveu–Schwarz* and *Ramond/Ramond*. (This is not a joke either and moreover illustrates the multiplicative nature of the spin structures.)

Given a spin structure $\mathrm{Spin}(M) \to M$ we can now construct **spinor bundles** as associated vector bundles. Let $S(s,t)$ denote a spinor representation of $\mathrm{Spin}(s,t)$ and define $S(M) \to M$ to be the vector bundle with total space

$$S(M) = \left(\mathrm{Spin}(M) \times S(s,t)\right) \Big/ \mathrm{Spin}(s,t) .$$

Depending on signature we might also have **half-spinor bundles** $S_\pm(M)$ associated to the half-spinor representations $S(s,t)_\pm$.

FIXME: I am not very happy with this lecture. I will eventually update this to include a small discussion of Čech cohomology with coefficients in a sheaf, to allow me at the very least to use the language freely.