Spin Geometry 2010

Tutorial Sheet 1

(Harder problems are adorned by a ✰.)

Problem 1.1.✰ Define a *Poisson algebra* as a vector space P with two bilinear operations $P \times P \to P$:

1. a commutative, associative multiplication written simply xy, and
2. a Lie bracket $[x, y]$,

subject to the compatibility condition

$$[x, yz] = [x, y]z + y[x, z] \ .$$

1. Show that an equivalent definition is a vector space P with a bilinear operation $P \times P \to P$, written $x \bullet y$, satisfying the following identity:

$$x \bullet (y \bullet z) = (x \bullet y) \bullet z - \tfrac{1}{3}\big((x \bullet z) \bullet y + (y \bullet z) \bullet x - (y \bullet x) \bullet z - (z \bullet x) \bullet y\big) \ .$$

2. State and prove a "super" version of the above result.

(*Hint*: the symmetric and skewsymmetric parts of $x \bullet y$ will be (proportional to) xy and $[x, y]$, respectively.)

Problem 1.2. Let $0 = F^{-1}A \subset F^0A \subset F^1A \subset \cdots \subset F^\infty A = A$ be a filtered associative algebra and let $\operatorname{Gr}^\bullet FA = \bigoplus_{k\geq 0} \operatorname{Gr}^k FA$, where $\operatorname{Gr}^k FA = F^kA/F^{k-1}A$ be the associated graded algebra.

1. Show that $\operatorname{Gr}^\bullet FA$ is commutative if and only if for all $a \in F^pA$ and $b \in F^qA$, $ab - ba \in F^{p+q-1}A$.
2. Let $\operatorname{Gr}^\bullet FA$ be commutative. Show that if $\bar{a} = a \mod F^{p-1}A$ and $\bar{b} = b \mod F^{q-1}A$ are elements in $\operatorname{Gr}^p FA$ and $\operatorname{Gr}^q FA$, respectively, then

   $$[\bar{a}, \bar{b}] = ab - ba \mod F^{p+q-2}$$

   is a Lie bracket and together with the commutative multiplication makes $\operatorname{Gr}^\bullet FA$ into a Poisson algebra.
3. State and prove a "super" version of the above two results.
4. For the Clifford algebra $C\ell(V, Q)$ with the filtration given in the lectures and $\Lambda^\bullet V$ its associated graded algebra, show that the Poisson bracket on $\Lambda^\bullet V$ is such that if $x, y \in V = \Lambda^1 V$, then

   $$[x, y] = -2B(x, y) \tag{1}$$

   and that the bracket is uniquely determined from this via the derivation property

   $$[\alpha, \beta \wedge \gamma] = [\alpha, \beta] \wedge \gamma + (-1)^{|\alpha||\beta|}\beta \wedge [\alpha, \gamma] \ ,$$

   where α, β are any two homogeneous elements of ΛV.

Problem 1.3. Let $\mathscr{D}(\mathbb{R}^n)$ denote the algebra of smooth differential operators on $\mathbb{R}^n$ under composition. Show that it is filtered by the order of the differential operator. Show that the associated graded algebra is the algebra of smooth functions on $T^*\mathbb{R}^n$ which are polynomial in the momenta (i.e., in the fibre coordinates). (The map from the filtered algebra to the associated graded algebra is called the *principal symbol*.) Show that the Poisson bracket which is induced by the commutator of differential operators agrees with the standard Poisson bracket on $T^*\mathbb{R}^n$ (restricted to the smooth functions which are polynomial in the momenta).

Problem 1.4. Show that $C\ell(3,0) \cong \mathbb{H} \oplus \mathbb{H}$ without using the periodicity results of the second lecture.

Problem 1.5. For each of the low-dimensional Clifford algebras C in the first lecture, describe the even subalgebra C_0 as a real (ungraded) associative algebra.

Problem 1.6.☆ Let (V, Q) be a real n-dimensional quadratic vector space with Q nondegenerate and choose a (pseudo)orthonormal basis e_i for V such that $Q(e_i) = \varepsilon_i$, where $\varepsilon_i^2 = 1$. Let Γ_i be the corresponding elements of $C\ell(V, Q)$. Show that the 2^{n+1} elements $\pm\mathbf{1}, \pm\Gamma_i, \pm\Gamma_{ij} (i < j), \dots, \pm\Gamma_{12\cdots n}$ define a finite group Γ under Clifford multiplication. Show that $C\ell(V, Q)$ is the quotient of the group algebra $\mathbb{R}\Gamma$ by 2-sided ideal generated by $(-\mathbf{1}) + \mathbf{1}$. (Notice that $\mathbf{1}$ and $-\mathbf{1}$, Γ_i and $-\Gamma_i$, et cetera, are *different* elements of Γ and quotienting by the ideal imposes that $-\mathbf{1}$ is indeed equal to $(-1)\mathbf{1}$.) This means that the Clifford algebra is almost a group algebra. Use this to show that any finite-dimensional representation of $C\ell(V, Q)$ is unitarizable.