Spin Geometry 2010

Tutorial Sheet 5

(Harder problems, if any, are adorned with a ✰.)

Problem 5.1. Let $P \to M$ be a principal G-bundle over M and let $E = P \times_G F \to M$ denote the associated vector bundle defined by a representation $\varrho : G \to GL(F)$ of G on a vector space F. Fill in the details of the proof of the graded $C^\infty(M)$-module isomorphism

$$\Omega^\bullet_G(P, F) \cong \Omega^\bullet(M, E)$$

between basic differential forms on P with values in F and differential forms on M with values in E for all k.

Problem 5.2. Let $\{U_\alpha\}$ be a trivialising cover for a principal G-bundle $\pi : P \to M$ and let $\{s_\alpha\}$ denote the corresponding local sections. Let ω be a connection 1-form on P and let $A_\alpha = s_\alpha^* \omega$ denote the corresponding gauge fields. Prove that for all $m \in U_{\alpha\beta}$,

$$A_\alpha(m) = g_{\alpha\beta}(m) A_\beta(m) g_{\alpha\beta}(m)^{-1} - d g_{\alpha\beta} g_{\alpha\beta}^{-1} , \tag{1}$$

where $g_{\alpha\beta} : U_{\alpha\beta} \to G$ are the transition functions of the bundle. Conversely, given gauge fields A_α subject to equation (1) on overlaps, define

$$\omega_\alpha = \mathrm{Ad}_{g_\alpha^{-1}} \circ \pi^* A_\alpha + g_\alpha^{-1} d g_\alpha , \tag{2}$$

and show that ω_α is the restriction to $\pi^{-1} U_\alpha$ of a connection 1-form on P.

Problem 5.3. Verify that the local expression for the covariant derivative in terms of gauge fields is indeed covariant.

Problem 5.4. Prove that the curvature tensor of the Levi-Civita connection on a riemannian manifold $(M.g)$ is indeed a tensor. Prove all the identities of the curvature tensor and in addition prove that

$$g(R(X,Y), Z, W) = g(R(Z,W), X, Y)$$

for all $X, Y, Z, W \in \mathscr{X}(M)$ and conclude that the Ricci tensor is symmetric. Finally, prove that formula for the decomposition of the Riemann curvature tensor:

$$R = \frac{s}{2n(n-1)} g \odot g + \frac{1}{n-2}(r - \frac{s}{n} g) \odot g + W$$

in terms of the Weyl curvature tensor W, the Ricci tensor r and the curvature scalar s.

Problem 5.5. Prove that the local expression given in the notes

$$\mathscr{E}^* \omega = \tfrac{1}{2} \sum_{i,j} g(\nabla e_i, e_j) e^i \curlywedge e^j$$

for the gauge field corresponding to the Levi-Civita connection of a riemannian manifold (M, g) is correct, by interpreting the tangent bundle TM as an associated vector bundle of the orthonormal frame bundle O(M) and showing that the covariant derivative $d + \mathscr{E}^* \omega$ is metric and torsion-free.

Problem 5.6. Show that the curvature 2-form of the Clifford-valued gauge field

$$\tfrac{1}{4}\sum_{i,j} g(\nabla e_i, e_{)} e^i e^j$$

is given by

$$\tfrac{1}{4}\sum_{i,j} \Omega_{ij} e^i e^j$$

where $\Omega_{ij}(X,Y) = g(R(X,Y)e_i, e_j)$ for all $X, Y \in \mathscr{X}(M)$. Prove that Clifford-valued covariant derivative is compatible with the Clifford action of ΛTM on any bundle of Clifford-modules:

$$\nabla_X(\theta \cdot \psi) = \nabla_X \theta \cdot \psi + \theta \cdot \nabla_X \psi \ ,$$

for all $\theta \in \Lambda TM$, ψ a pinor field and $X \in \mathscr{X}(M)$.

Problem 5.7.☆ Describe the Dirac monopole (including the "Dirac string") in the language of principal fibre bundles.