Schrödinger holography

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Introduction

- Gauge/gravity dualities have become an important new tool in extracting strong coupling physics.
- The best understood examples of such dualities involve relativistic quantum field theories.
- Strongly coupled non-relativistic QFTs are common place in condensed matter physics and elsewhere.
- It is natural to wonder whether holography can be used to obtain new results about such non-relativistic strongly interacting systems.

The non-relativistic conformal group

In non-relativistic physics the Poincaré group is replaced by the Galilean group. It consists of

- the temporal translation $\mathcal{H}$, spatial translations $\mathcal{P}^i$, rotations $\mathcal{M}^{ij}$, Galilean boosts $\mathcal{K}^i$ and the mass operator $\mathcal{M}$.

The conformal extension adds to these generators

- the non-relativistic scaling operator $\mathcal{D}$ and the non-relativistic special conformal generator $\mathcal{C}$.

The scaling symmetry acts as

$$t \to \lambda^2 t, \qquad x^i \to \lambda x^i$$

Schrödinger group

- This is the maximal kinematical symmetry group of the free Schrödinger equation [Niederer (1972)], hence its name: Schrödinger group $Sch(d)$.

Interacting systems that realize this symmetry include:

- Non-relativistic particles interacting through an $1/r^2$ potential.
- Fermions at unitarity. (Fermions in three spatial dimensions with interactions fine-tuned so that the s-wave scattering saturates the unitarity bound).
  This system has been realized in the lab using trapped cold atoms [O'Hara et al (2002) ...] and has created enormous interest.

Schrödinger with general exponent z

- One can also add to the Galilean generators (including the mass $\mathcal{M}$) a generator of dilatations $\mathcal{D}_z$ acting as

$$t \to \lambda^z t, \qquad x^i \to \lambda x^i$$

  but for general z there is no special conformal symmetry.
- This algebra will be denoted as $Sch_D(z)$.
- Removing the central term $\mathcal{M}$ gives the symmetries of a D-dimensional Lifshitz theory with exponent z, denoted $Lif_D(z)$.

Holographic realization

Holographically these symmetry groups should be realized as isometries of the dual spacetimes.

For example, Anti-de Sitter in $(D+1)$ dimensions admits as an isometry group the D-dimensional conformal group $SO(D,2)$.

Holography for Schrödinger

[Son (2008)] and [K. Balasubramanian, McGreevy (2008)] initiated a discussion of holography for $(d+1)$ dimensional spacetimes with metric,

$$ds^2 = -\frac{b^2 du^2}{r^4} + \frac{2dudv + dx^i dx^i + dr^2}{r^2},$$

- When $b = 0$ this is the AdS_{d+1} metric.
- This metric realizes geometrically the Schrödinger group in $D = (d-1)$ dimensions.
- In order for the mass operator $\mathcal{M}$ to have discrete eigenvalue lightcone coordinate v must be compactified with $u \to t$.

Holography for general z Schrödinger

More generally one can also realize $Sch_D(z)$ geometrically in $(d+1) = (D+2)$ dimensions via

$$ds^2 = \frac{\sigma^2 du^2}{r^{2z}} + \frac{2dudv + dx^i dx^i + dr^2}{r^2},$$

- The dual field theory is then d-dimensional, with anisotropic scale invariance $u \to \lambda^z u$, $v \to \lambda^{2-z} v$ and $x^i \to \lambda x^i$.
- Various CMT models of this type e.g. Cardy's continuum limit of chiral Potts model ($z = 4/5$).
- The theory becomes a non-relativistic theory in D dimensions upon compactifying v or u.
- As we will see, this reduction is always a null compactification, regardless of values of (z, σ).

Lifshitz spacetimes

The Lifshitz symmetry $Lif_D(z)$ may be realized geometrically in $(D+1)$ dimensions [Kachru et al, 2008]

$$ds^2 = \frac{dr^2}{r^2} - \frac{dt^2}{r^{2z}} + \frac{dx^i dx_i}{r^2}.$$

- The radial direction is again associated with scale transformations.
- The holographic realization of Lifshitz is more conventional c.f. Schrödinger cases where the mass generator is geometrically realized via extra dimensions.

Bulk systems

These metrics solve the field equations for e.g.

- Gravity coupled to **massive vectors**
- **Topologically massive gravity (TMG) in 3d**

In the latter case the solution with $z = 2$ was called "null warped AdS_3" and conjectured to be dual to a $2d$ CFT with certain (c_L, c_R) [Anninos et al (2008)].

→ This is a rather different proposal for the physics of the solution.

The key issues

- These spacetimes are not asymptotically AdS and so the usual holographic set up is not automatically applicable.

Even basic issues such as:

- is the dual theory a local QFT?
- what is the correspondence between bulk fields and dual operators?

are not well understood.

The issues

To avoid the complications of a null compact direction, we consider the spacetime with v *non-compact*.
The main features of the Schrödinger duality are:

- The dual theory is a deformation of a d-dimensional CFT.
- The deformation is **irrelevant** w.r.t. relativistic conformal group.
- The deformation is **exactly marginal** w.r.t. **non-relativistic** conformal group.
- For $z = 2$ the theory becomes non-local in the v direction.

References

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Plan

1. **Weak chirality limit and field theory deformations**
2. Holographic dictionary for probe operators
3. The stress energy tensor sector
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The small b "weak chirality" limit

In the small b limit the geometry

$$ds^2 = -\frac{b^2 du^2}{r^{2z}} + \frac{2dudv + dx^i dx^i + dr^2}{r^2},$$

is a small perturbation of AdS and standard AdS/CFT applies.

Massive vector model

- **Massive vector** model. Geometry solves equations of motion of:

$$S = \int d^{d+1}x \sqrt{-G}(R - 2\Lambda - \frac{1}{4}F_{\mu\nu}F^{\mu\nu} - \frac{1}{2}m^2 A_\mu A^\mu)$$

  with $m^2 = z(d + z - 2)$ and vector field

$$A_u = \frac{b}{r^z}.$$

- Consistent truncations include additional scalar fields, but these will not play a role here.

Massive vector model

- Working to linear order in b, background corresponds to a field theory deformation:

$$S_{CFT} \to S_{CFT} + \int d^d x \, b^i X_i$$

  - $\to$ X_i has dimension $(d + z - 1)$ and is dual to the bulk vector field.
  - $\to$ b^i is a null vector with only non-zero component $b^v = b$.

Topologically massive gravity (TMG)

- **TMG** equation of motion

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R + \Lambda g_{\mu\nu} + \frac{1}{2\mu}\left(\epsilon_{\mu}{}^{\rho\sigma}\nabla_{\rho} R_{\sigma\nu} + \epsilon_{\nu}{}^{\rho\sigma}\nabla_{\rho} R_{\sigma\mu}\right) = 0.$$

  is third order and chiral.
- The additional boundary condition (cf Einstein gravity) is related to a new dual CFT operator X. (van Rees, Skenderis, M.T. 2009)
- Dual CFT contains both T_{ij} and the tensor operator X.

- AdS/CFT dictionary at small b^2 implies:

$$S_{CFT} \to S_{CFT} + \int d^2x \, b^{ij} X_{ij}$$

  $\rightarrow$ X_{ij} has dimension $(z+1, z-1)$.
  $\rightarrow$ b^{ij} is a null tensor with only non-zero component $b^{vv} = -b^2$.

- A priori b^2 can have either sign, but $b^2 < 0$ for black hole solutions and $b^2 > 0$ for stability.

Scale invariance

- The deforming operators are relevant for $z < 1$ and irrelevant for $z > 1$, with respect to relativistic dilatations.
- In all cases however the non-relativistic scaling dimension of the deforming operator is

$$\Delta_s = d$$

  and so the deformations are marginal wrt anisotropic scaling symmetry with exponent z!
- Next we need to understand what happens at finite b, focus first on $z = 2$ case.

Finite b

Bulk perspective:

- Schrödinger solutions solve the complete non-linear equations.

$\rightarrow$ The theory is Schrödinger invariant for any b.

Boundary QFT perspective:

- We analyzed this question using conformal perturbation theory.

$\rightarrow$ The deforming operator is indeed exactly marginal wrt Schrödinger.

Exact marginality

To explain this computation we need a few facts about theories with Schrödinger invariance:

- Operators are labeled by their non-relativistic scaling dimension, Δ_s and their charge under $\mathcal{M}$, the mass operator.
- In our context the mass operator is the lightcone momentum k_v.
- Operators with different k_v are considered as independent operators.
- In our case, the deforming operator has zero lightcone momentum, $k_v = 0$.

Exact marginality

To prove that the operator is exactly marginal it suffices to show that its 2-point function does not receive any corrections when we turn on b.

$$\langle X_v(k_v{=}0, u_1, x_1^i) X_v(k_v{=}0, u_2, x_2^i)\rangle_{\mathbf{b}} = \langle X_v(k_v{=}0, u_1, x_1^i) X_v(k_v{=}0, u_2, x_2^i)\rangle_{\mathbf{b=0}}$$

This can be studied using conformal perturbation theory.

Conformal perturbation theory

One can show that

$$\langle X_v(k_v) \prod_{i=1}^{n} b^\mu \cdot X_\mu(k_v{=}0) X_v(-k_v) \rangle_{\rm CFT} = \langle X_v(k_v) X_v(-k_v) \rangle_{\rm CFT} \, (b^v k_v)^n f(\log k_v, \ldots)$$

where $f(\log k_v, \ldots)$ is a dimensionless function that depends at most polynomially on $\log k_v$.

- Taking the limit $k_v \to 0$, establishes that $X_v(k_v{=}0)$ is exactly marginal.
- The dimensions of operators with $k_v \neq 0$ receive corrections,

$$\Delta_s = \Delta_s(b = 0) + \sum_{n>0} \mathbf{c_n} (b k_v)^n$$

Schrödinger summary

- We started with a relativistic CFT and deformed it by an irrelevant operator which is however exactly marginal from the perspective of the Schrödinger group.
- This is a general procedure to generate novel, anisotropic scale invariant theories.

- For general z there is a similar classification of marginal deforming operators X

$$S_{CFT} \rightarrow S_{CFT} + \int d^d x bX$$

which preserve the chiral scale invariance.

- Certain features depend on the value of z, e.g. unless $z = 2n$ operators acquire no k_v dependent anomalous dimensions....

The case of $z = 0$

- The case of $z = 0$ (which is still asymptotically AdS)

$$ds^2 = \frac{dr^2}{r^2} + \sigma^2 du^2 + \frac{1}{r^2}(2dudv + dx^i dx_i)$$

is interesting because of its relation to $z_L = 2$ Lifshitz upon dimensional reduction [Donos and Gauntlett]

$$ds^2 = \frac{dr^2}{r^2} + \sigma^2 (du + \frac{dv}{\sigma^2 r^2})^2 - \frac{dv^2}{\sigma^2 r^4} + \frac{dx^i dx_i}{r^2}.$$

One can realize this via a scalar operator deformation, with chiral source, but note that u is null, so DLCQ needed to obtain Lifshitz.

Lifshitz holography

- An important open question has been how to embed Lifshitz geometries into string theory.
- The best understood such embedding (unfortunately) relates Lifshitz to a DLCQ of a deformed CFT.

Discrete Lightcone Quantization (DLCQ)

- To obtain a non-relativistic system we need to compactify the v direction (for $z > 1$) or the u direction (for $z < 1$).

But periodically identifying a null circle is subtle!

- The zero mode sector is usually problematic (and here the problem is seen in ambiguities in the initial value problem in the spacetime).
- Strings winding the null circle become very light.

As we will see later, operators associated with the extra null direction also contaminate the physics in the reduced theory (e.g. peculiar hydrodynamics).

Plan

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Spectrum of deformed Schrödinger theory

- The next question is then to understand the spectrum of operators at the new fixed point.
- We have seen how in conformal perturbation theory at small b the non-relativistic dimension Δ_s of operators with $k_v \neq 0$ changes as we go from one fixed point to the other.
- We will analyze this question from the bulk perspective, where the deformation parameter b is finite.

Probe scalar

Let us consider a probe scalar field in the 3d Schrödinger background,

$$S = -\frac{1}{2} \int d^3x \sqrt{-g} \Big(\partial_\mu \Phi \partial^\mu \Phi + m^2 \Phi^2 \Big).$$

The field equations are

$$\ddot{\Phi} + 2\dot{\Phi} + \Box_\zeta \Phi - (m^2 - b^2 \partial_v^2) \Phi = 0$$

The asymptotics of the solution are

$$\Phi = e^{(\Delta_s - 2)y} \Big(\phi_{(0)}(k) + \ldots + e^{-(2\Delta_s - 2)y} \phi_{(2\Delta_s - 2)}(k) + \ldots \Big)$$

with $r = e^{-y}$.

Probe scalar

- The dual operator has dimension

$$\Delta_s = 1 + \sqrt{1 + m^2 + b^2 k_v^2}$$

- For small b it takes the form we found earlier using conformal perturbation theory

$$\Delta_s = \Delta_s(b = 0) + \sum c_n (b k_v)^n$$

where $\Delta_s(b = 0) = 1 + \sqrt{1 + m^2}$ is the standard holographic formula for the dimension of a scalar operator.

- Square root form is generic to all holographic realizations, but does not follow from Schrödinger invariance alone.

Correlation functions

- To compute correlation functions we need to compute the on-shell value of the action.
- This suffers from the usual infinite volume divergences.
- Adapting holographic renormalization we find that we need counterterms

$$S_{\text{ct},\Delta_s \lesssim 3} = -\frac{1}{2}\int d^2k\,\sqrt{-\zeta}\Big((\Delta_s - 2)\Phi^2 + \frac{k_\zeta^2\Phi^2}{2\Delta_s - 4}\Big)$$

- When $b = 0$ these reduce to the counterterms for the scalar field in AdS.

Non-locality

$$S_{\mathrm{ct},\Delta_s \lesssim 3} = -\frac{1}{2}\int d^2k\,\sqrt{-\zeta}\Big((\Delta_s - 2)\Phi^2 + \frac{k_\zeta^2\Phi^2}{2\Delta_s - 4}\Big)$$

- Because Δ_s depends on k_v, the counterterms are not polynomials in k_v.
- The theory is non-local in the v direction.

2-point function

- Having determined the counterterms, the 2-point function can now be extracted from an exact solution of the linearized field equations[1]:

$$\langle \mathcal{O}_{\Delta_s}(u, k_v)\mathcal{O}_{\Delta_s}(0, -k_v)\rangle = c_{\Delta_s,k_v}\delta_{\Delta,\Delta_s}u^{-\Delta_s},$$

  where c_{Δ_s,k_v} is a (specific) normalization factor.
- This is precisely of the expected form for a 2-point function of a Schrödinger invariant theory [Henkel (1993)].

[1] Real-time issues considered in [Leigh-Hoang, Blau et al (2009)]

Holographic dictionary for probe operators

Renormalized correlation functions can be computed from perturbing around Schrödinger and using holographic renormalization, provided that we allow for non-locality in the v direction.

Holographic dictionary for probe operators

- For $z \leq 2$ one finds maps between operator expectation values and coefficients in the asymptotic expansion

$$\langle \mathcal{O}_{\Delta_s} \rangle \sim \phi_{\Delta_s} + f(\phi_{(0)}),$$

but counterterms respect only the anisotropic symmetry of the dual theory, and can be non-local in v.

- Matches boundary field theory analytic structure!
- Correlation functions are (in $d = 2$)

$$\langle \mathcal{O}_{\Delta_s}(u, v) \mathcal{O}_{\Delta_s}(0, 0) \rangle = \frac{1}{u^{\Delta_s}} F\left(\frac{u^{2-z}}{v^z}\right).$$

F is a priori an arbitrary function, whilst holographically only specific universal functions F appear.

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The stress energy tensor sector

- For asymptotically locally *AdS* spacetimes, near the conformal boundary

$$ds^2 = \frac{dr^2}{r^2} + \frac{1}{r^2} g_{ij}(x,r) dx^i dx^j$$

Expanding

$$g_{ij} = g_{(0)ij} + \cdots + r^d g_{(d)ij} + \cdots$$

the expectation value of the dual stress energy tensor sourced by $g_{(0)ij}$ is

$$\langle T_{ij} \rangle = g_{(d)ij} + X_{ij}[g_{(0)}]$$

and characterizes the state in the dual CFT.

Asymptotically locally Schrödinger?

- How to define asymptotically locally Schrödinger for metric and matter fields, i.e. for

$$ds^2 = \frac{dr^2}{r^2} + \frac{1}{r^2} g_{ij}(x,r) dx^i dx^j$$

  what is the appropriate behavior for $g_{ij}(x,0)$, and for the matter?
- What are the operators dual to the metric and matter fields?
- What is the explicit map between these operators and the bulk field asymptotics?

This is very non-trivial for all $z > 1$ cases, since they are not asymptotically locally AdS.

Gravitational sector

To illustrate the issues, it is useful to first consider linearized perturbations

$$ds^2 = \frac{dr^2}{r^2} + \frac{2dudv}{r^2} - b^2\frac{du^2}{r^4} + \frac{1}{r^2}h_{ij}dx^i dx^j$$

around the Schrödinger background (in 3d).

- Both models (massive vector and TMG) admit orthogonal sets of solutions to their linearized equations:
  - The 'T' solutions are associated with the dual stress energy tensor.
  - The 'X' solutions are associated with the dual deforming operator.

'X' solutions: TMG

- These propagating fluctuations satisfy a hypergeometric equation.
- The dimension of the dual operator is

$$\Delta_s(X_{vv}) = 1 + \sqrt{1 + b^2 k_v^2}$$

  This is marginally irrelevant, and has the correct limit found in the field theory as $b \to 0$.
- The linearized solution is more singular at the boundary than the Schrödinger background. This is due to the fact that the operators with $k_v \neq 0$ are irrelevant.
- The 2-point function takes the Schrödinger form for an operator of this dimension.

‘T’ solutions

The ‘T’ mode metric perturbations take the form:

$$h^T_{uu} = \frac{1}{r^2} h_{(-2)uu} + \tilde{h}_{(0)uu} \log(r^2) + h_{(0)uu} + r^2 h_{(2)uu}$$

$$h^T_{uv} = \frac{1}{r^2} h_{(-2)uv} + \tilde{h}_{(0)uv} \log(r^2) + h_{(0)uv} + r^2 h_{(2)uv}$$

$$h^T_{vv} = h_{(0)vv} + r^2 h_{(2)vv},$$

- These modes at $b = 0$ reduce to the modes that couple to the energy momentum tensor, T_{ij}.
- The general solution is more singular as $r \to 0$ than the Schrödinger background, since certain components of the stress energy tensor are irrelevant wrt Schrödinger.
- [Son] set $h_{vv} = 0$, and hence switched off and constrained dual stress energy tensor.

Stress energy tensor

Subtleties in understanding this sector:

- In a **non-relativistic theory** the tensor t_{ij} that contains the conserved energy and momentum is not symmetric and therefore cannot couple to any metric mode.
- This tensor t_{ij} couples instead to the vielbein $\rightarrow$ natural to formulate holography as a Dirichlet problem for the vielbein.
- Part of stress energy tensor is irrelevant, so sources must be treated perturbatively.

A long story....

Linearized level

- Treating $e_{(0)}$ as the sources, one can renormalize the bulk action using counterterms with only allowed non-locality in the v direction.
- We obtain maps between operators and asymptotic data, the expected anomalous Ward identities e.g. for TMG

$$\langle t_{uv} \rangle + b^2 \langle X_{vv} \rangle = \mathcal{A}[e_{(0)}]$$

- Varying the renormalized action and using the regular solutions of the linearized equations gives us two point functions for t_{ij} and the operators X.

This completes the analysis at the linearized level.

Stress energy tensor

Going beyond the linearized analysis, the issues are:

- Asymptotically locally Schrödinger?
- Reduction along v?
    - The operator t_{ij} contains not just the $(d-1)$-dimensional energy current, mass current and stress tensor.
    - A key use of holography would be hydrodynamics of the $(d-1)$-dimensional relativistic theory, but the $(d-1)$-dimensional stress energy is not conserved!

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Conclusions

The dual to $z = 2$ Schrödinger and "null warped" backgrounds is

- a deformation of a d-dimensional CFT.
- The deformation is **irrelevant** w.r.t. relativistic conformal group.
- The deformation is **exactly marginal** w.r.t. **non-relativistic** conformal group.
- The theory is non-local in the v direction.

Analogous story for dynamical exponents $z \neq 2$.

Schrödinger phenomenology: a generic prediction

- In the bulk geometries the deformation parameter b can take any value.

The physical systems being modeled should have a corresponding parameter, adjusting which preserves the Schrödinger scale invariance.

- In the $(D + 1)$-dimensional theory (before null reduction) this should be a "chiral" interaction which can be arbitrarily weak or strong.

Schrödinger summary

Geometric realization of the mass generator $\mathcal{M}$ of the Schrödinger algebras is undesirable, and inevitably leads to the dual theory being a DLCQ of a deformed CFT.

Perhaps these deformed theories with anisotropic scale invariance are physically interesting without DLCQ, e.g. Cardy's chiral Potts model?

Null dipole theory

- [Maldacena et al, Herzog et al (2008)] argued that the massive vector case in $d = 4$ is dual to a null dipole theory, a non-local deformation of $N = 4$ SYM.
- In the null dipole theory, the ordinary product is replaced by a non-commutative product that depends on a null vector [Ganor et al (2000)]. Expressed in terms of ordinary products the null dipole theory contains terms that are:
  - irrelevant from the relativistic CFT point of view
  - marginal from the Schrödinger perspective

  $\rightarrow$ Null dipole is a specific type of Schrödinger theory.

Open questions

- Very little is currently known about null dipole theories: gauge invariant operators? divergence structure the same as we found in gravity?

- TMG admits extremal null warped black hole solutions which are asymptotically Schrödinger

$$ds^2 = \frac{dr^2}{r^2} + du^2\left(\frac{1}{r^4} + \frac{1}{r^2} + \alpha^2\right) + \frac{2}{r^2} du dv,$$

  in which $T_L = T_H = 0$ and $T_R = \alpha/\pi$.
- Anninos et al used thermal Cardy formula $S = \frac{1}{3}\pi^2 c_R T_R$ to account for black hole entropy.

- However, the dual theory is actually a $z = 2$ anisotropic deformation of a CFT:

$$S_{CFT} \rightarrow S_{CFT} + \int d^2x \, X_{vv}$$

  so how does the anisotropic theory reproduce black hole entropy?

- It turns out that a Cardy formula is inherited in these deformed theories.... cf [Dijkgraaf, 1996]

Future directions

Extension to other dualities:

Kerr/CFT?

Warped *AdS* spaces arise in NHEK: is the dual theory actually a deformation of a CFT of the type we discussed? [Guica and Strominger]

More importantly, for CMT applications, given that the dual theory is

$$S_{\mathrm{CFT}} \to S_{\mathrm{CFT}} + b \int d^d x X$$

Chiral deformation?

Is there a physical interpretation of the deforming operator X in cold atom systems?