Alternating Sign Matrices and Descending Plane Partitions

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Introduction

- **Plane Partitions** were introduced by Mac Mahon about a century ago. However **Descending Plane Partitions** (DPPs), as well as other variations on plane partitions (symmetry classes), were considered in the 80s. [Andrews]

- **Alternating Sign Matrices** (ASMs) also appeared in the 80s, but in a completely different context, namely in Mills, Robbins and Rumsey’s study Dodgson’s condensation algorithm for the evaluation of determinants.

- One of the possible formulations of the **Alternating Sign Matrix conjecture** is that these objects are in bijection (for every size $n$). (proved by Zeilberger in ’96 in a slightly different form)
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Interest in the mathematical physics community because of

1. Kuperberg’s alternative proof of the Alternating Sign Matrix conjecture using the connection to the six-vertex model. (’96)

2. The Razumov–Stroganov correspondence and related conjectures. (’01)

A proof of all these conjectures would probably give a fundamentally new proof of the ASM (ex-)conjecture.

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T. Fonseca and P. Zinn-Justin: proof of the doubly refined Alternating Sign Matrix conjecture ('08).
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Today’s talk is about the proof of another generalization of the ASM conjecture formulated in ’83 by Mills, Robbins and Rumsey.
Iterative use of the Desnanot–Jacobi identity:

\[
\begin{array}{|c|c|} \hline
\begin{array}{|c|c|} \hline
\end{array} & \begin{array}{|c|} \hline
\end{array} \\
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\end{array} = \begin{array}{|c|c|c|} \hline
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\end{array} & \begin{array}{|c|} \hline
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\end{array} - \begin{array}{|c|c|c|} \hline
\begin{array}{|c|c|} \hline
\end{array} & \begin{array}{|c|} \hline
\end{array} & \begin{array}{|c|} \hline
\end{array} \\
\hline
\end{array}
\]

allows to compute the determinant of a $n \times n$ matrix by computing the determinants of the connected minors of size 1, \ldots, $n$.

What happens when we replace the minus sign with an arbitrary parameter?
Iterative use of the Desnanot–Jacobi identity:

\[
\begin{array}{|c|c|c|c|}
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\times & \times & \times & \times \\
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\end{array}
\quad - \quad
\begin{array}{|c|c|c|c|}
\hline
\times & \times & \times & \times \\
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\[
\begin{array}{cccc}
\square & \square & \square & \square \\
\square & \square & \square & \square \\
\square & \square & \square & \square \\
\lambda & \square & \square & \square \\
\end{array}
\]

allows to compute the determinant of a $n \times n$ matrix by computing the determinants of the connected minors of size $1, \ldots, n$.

What happens when we replace the minus sign with an arbitrary parameter?
Theorem (Robbins, Rumsey, '86)

If $M$ is an $n \times n$ matrix, then

$$\det_{\lambda} M = \sum_{A \in \text{ASM}(n)} \lambda^{\nu'(A)} (1 + \lambda)^{\mu(A)} \prod_{i,j=1}^{n} M^{A}_{ij}$$

Here $\text{ASM}(n)$ is the set of $n \times n$ Alternating Sign Matrices, that is matrices such that in each row and column, the non-zero entries form an alternation of $+1$s and $-1$s starting and ending with $+1$. 
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For $n = 3$, there are 7 ASMs:

\[
ASM(3) = \left\{ \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & -1 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \right\}
\]
$\mu(A)$ is the number of $-1$s in $A$.

$\nu'(A)$ is a generalization of the inversion number of $A$:

$$\nu'(A) = \sum_{1 \leq i < i' \leq n, 1 \leq j' < j \leq n} A_{ij} A_{i'j'}$$

In what follows it is more convenient to consider another generalization of the inversion number, namely

$$\nu(A) = \nu'(A) - \mu(A) = \sum_{1 \leq i < i' \leq n, 1 \leq j' < j \leq n} A_{ij} A_{i'j'}$$

Finally, for future purposes define $\rho(A)$ to be the number of $0$'s to the left of the $1$ in the first row of $A$. 
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Finally, for future purposes define \( \rho(A) \) to be the number of 0’s to the left of the 1 in the first row of \( A \).
A **Descending Plane Partition** is an array of positive integers ("parts") of the form

\[
\begin{array}{cccc}
D_{11} & D_{12} & \cdots & D_{1,\lambda_1} \\
D_{22} & \cdots & \cdots & D_{2,\lambda_2+1} \\
\vdots & \ddots & \ddots & \ddots \\
D_{tt} & \cdots & D_{t,\lambda_t+t-1}
\end{array}
\]

such that

- The parts decrease weakly along rows, i.e., \( D_{ij} \geq D_{i,j+1} \).
- The parts decrease strictly down columns, i.e., \( D_{ij} > D_{i+1,j} \).
- The first parts of each row and the row lengths satisfy

\[
D_{11} > \lambda_1 \geq D_{22} > \lambda_2 \geq \cdots \geq D_{t-1,t-1} > \lambda_{t-1} \geq D_{tt} > \lambda_t
\]
Let $\text{DPP}(n)$ be the set of DPPs in which each part is at most $n$, i.e., such that $D_{ij} \in \{1, \ldots, n\}$.

Example

For $n = 3$, there are 7 DPPs:

$$\text{DPP}(3) = \left\{ \emptyset, \begin{array}{c} 3 \\ 2 \end{array}, 2, 3 \ 3, 3, 3 \ 2, 3 \ 1 \right\}$$
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$$DPP(3) = \left\{ \emptyset, \begin{array}{c} 3 \\ 2 \end{array}, 2, 3 3, 3, 3 2, 3 1 \right\}$$
Define statistics for each $D \in \text{DPP}(n)$ as:

$\nu(D) =$ number of parts of $D$ for which $D_{ij} > j - i$,

$\mu(D) =$ number of parts of $D$ for which $D_{ij} \leq j - i$,

$\rho(D) =$ number of parts equal to $n$ in (necessarily the first row of) $D$. 
DPP enumeration

Theorem (Andrews, 79)

*The number of DPPs with parts at most \( n \) is:*

\[
|DPP(n)| = \prod_{i=0}^{n-1} \frac{(3i + 1)!}{(n + i)!} = 1, 2, 7, 42, 429 \ldots
\]
The Alternating Sign Matrix conjecture

The following result was first conjectured by Mills, Robbins and Rumsey in ’82:

**Theorem (Zeilberger, ’96; Kuperberg, ’96)**

The number of ASMs of size \( n \) is

\[
|\text{ASM}(n)| = \prod_{i=0}^{n-1} \frac{(3i + 1)!}{(n + i)!} = 1, 2, 7, 42, 429 \ldots
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NB: A third family is also known to have the same enumeration as ASMs and DPPs: TSSCPPs. In fact, Zeilberger’s proof consists of a (non-bijective) proof of equienumeration of ASMs and TSSCPPs.
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A more general result was conjectured by Mills, Robbins and Rumsey in ’83:

**Theorem (Behrend, Di Francesco, Zinn-Justin, ’11)**

The sizes of \{A \in \text{ASM}(n) \mid \nu(A) = p, \mu(A) = m, \rho(A) = k\} and \{D \in \text{DPP}(n) \mid \nu(D) = p, \mu(D) = m, \rho(D) = k\} are equal for any \(n, p, m\) and \(k\).

Equivalently, if one defines generating series:

\[
Z_{\text{ASM}}(n, x, y, z) = \sum_{A \in \text{ASM}(n)} x^\nu(A) y^\mu(A) z^\rho(A)
\]

\[
Z_{\text{DPP}}(n, x, y, z) = \sum_{D \in \text{DPP}(n)} x^\nu(D) y^\mu(D) z^\rho(D)
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then the theorem states that \(Z_{\text{ASM}}(n, x, y, z) = Z_{\text{DPP}}(n, x, y, z)\).
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\]

\[
Z_{\text{ASM/DPP}}(3, x, y, z) = 1 + x^3z^2 + x + x^2z^2 + xz + x^2z + xyz
\]
Strategy: write the two generating series as determinants:

\[ Z_{\text{ASM}}(n, x, y, z) = \det M_{\text{ASM}}(n, x, y, z) \]
\[ Z_{\text{DPP}}(n, x, y, z) = \det M_{\text{DPP}}(n, x, y, z) \]

and transform one matrix into another by row/column manipulations.

In what follows, we only give the proof in the “unrefined” case \( z = 1 \).
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Let $\text{6VDW}(n)$ be the set of all configurations of the six-vertex model on the $n \times n$ grid with DWBC, i.e., decorations of the grid’s edges with arrows such that:

- The arrows on the external edges are fixed, with the horizontal ones all incoming and the vertical ones all outgoing.
- At each internal vertex, there are as many incoming as outgoing arrows.

The latter condition is the “six-vertex” condition, since it allows for only six possible arrow configurations around an internal vertex:

![Diagram of six possible arrow configurations around an internal vertex](image-url)
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Statistics

Statistics also have a nice interpretation in terms of the six-vertex model: if $A \in \text{ASM}(n) \leftrightarrow C \in 6\text{VDW}(n)$,

$$\mu(A) = \frac{1}{2} ((\text{number of vertices of type } c \text{ in } C) - n)$$

$$\nu(A) = \frac{1}{2} (\text{number of vertices of type } a \text{ in } C)$$
Define the six-vertex partition function of the six-vertex model with DWBC to be:

\[
Z_{6\text{VDW}}(u_1, \ldots, u_n; v_1, \ldots, v_n) = \sum_{C \in 6\text{VDW}(n)} \prod_{i,j=1}^{n} C_{ij}(u_i, v_j)
\]

where the \(u_i\) (resp. the \(v_j\)) are parameters attached to each row (resp. a column), and \(C_{ij}\) is the type of configuration at vertex \((i, j)\).

\[
a(u, v) = uq - \frac{1}{vq}, \quad b(u, v) = \frac{u}{q} - \frac{q}{v}, \quad c(u, v) = \left(q^2 - \frac{1}{q^2}\right)\sqrt{\frac{u}{v}}
\]
Based on Korepin’s recurrence relations for $Z_{6VDW}$, Izergin found the following determinant formula:

**Theorem (Izergin, ’87)**

$$Z_{6VDW}(u_1, \ldots, u_n; v_1, \ldots, v_n) \propto \frac{\det_{1 \leq i,j \leq n} \left( \frac{1}{a(u_i, v_j)b(u_i, v_j)} \right)}{\prod_{1 \leq i < j \leq n} (u_j - u_i)(v_j - v_i)}$$

Problem: what happens in the homogeneous limit $u_1, \ldots, u_n, v_1, \ldots, v_n \to r$?
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Problem: what happens in the homogeneous limit $u_1, \ldots, u_n, v_1, \ldots, v_n \to r$?
The "naive" homogeneous limit:

\[ Z_{6VDW}(r, \ldots, r; r, \ldots, r) \propto \det_{0 \leq i, j \leq n-1} \frac{\partial^{i+j}}{\partial u^i \partial v^j} \left( \frac{1}{a(u, v)b(u, v)} \right) \bigg|_{u, v=r} \]

\[ \propto \det_{0 \leq i, j \leq n-1} \frac{\partial^{i+j}}{\partial u^i \partial v^j} \left( \frac{1}{uv - q^2} - \frac{1}{uv - q^{-2}} \right) \bigg|_{u, v=r} \]
Define $L_{ij}$ to be the $n \times n$ lower-triangular matrix with entries $\binom{i}{j}$, and $D$ to be the diagonal matrix with entries $\left(\frac{qr-q^{-1}r^{-1}}{q^{-1}rqr^{-1}}\right)^i$, $i = 0, \ldots, n - 1$.

**Proposition (Behrend, Di Francesco, Zinn-Justin, ’11)**

$$Z_{6\text{VDW}}(r, \ldots, r; r, \ldots, r) \propto \det \left( I - \frac{r^2 - q^{-2}}{r^2 - q^2} DLDL^T \right)$$

Proof: write the determinant as $\det(A_+ - A_-)$, note that $A_\pm$ is up to a diagonal conjugation $\frac{1}{r^2 - q^\pm 2} D_\pm LD_\pm L^T$, pull out $\det A_+$ and conjugate $I - A_- A_+^{-1} \ldots$
Define $L_{ij}$ to be the $n \times n$ lower-triangular matrix with entries $\binom{i}{j}$, and $D$ to be the diagonal matrix with entries $\left( \frac{q r - q^{-1} r^{-1}}{q^{-1} r - q r^{-1}} \right)^i$, $i = 0, \ldots, n - 1$.

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Rewriting the previous proposition in terms of Boltzmann weights $a$, $b$, $c$, and then switching to $x = (a/b)^2$, $y = (c/b)^2$, we finally find $Z_{ASM}(n, x, y, 1) = \det M_{ASM}(n, x, y, 1)$ with

$$M_{ASM}(n, x, y, 1)_{ij} = (1 - \omega)\delta_{ij} + \omega \sum_{k=0}^{\min(i,j)} \binom{i}{k} \binom{j}{k} x^k y^{i-k}$$

with $i, j = 0, \ldots, n-1$ and $\omega$ a solution of

$$y\omega^2 + (1 - x - y)\omega + x = 0$$
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Equality of determinants

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Generalizations

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Generalizations

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The ASM-DPP conjecture
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Generalizations

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Generalizations

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Generalizations

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Statistics

Statistics also have a nice interpretation in terms of Nonintersecting lattice paths (NILPs):

\[ D = \]

\[ \nu(D) = 7 \]
\[ \mu(D) = 2 \]
NILPS are (lattice) free fermions:

Number of NILPs from $S_i$ to $E_i$, $i = 1, \ldots, n$

$$= \det_{i,j=1,\ldots,n} \text{(Number of (single) paths from } S_i \text{ to } E_j)$$

and similarly with weighted sums.
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and similarly with weighted sums.
Here we are also summing over endpoints and the number of paths ("grand canonical partition function"): 
\[ Z_{DPP}(n, x, y, 1) = \det M_{DPP}(n, x, y, 1) \]

with

\[ M_{DPP}(n, x, y, 1) = \delta_{ij} + \sum_{k=0}^{i-1} \sum_{\ell=0}^{\min(j,k)} \binom{j}{\ell} \binom{k}{\ell} x^{\ell+1} y^{k-\ell} \]

Note that the second term is a product of two discrete transfer matrices...
We have

\[(I - S)M_{\text{DPP}}(n, x, y, 1)(I + (\omega - 1)S^T)\]
\[= (I + (x - \omega y - 1)S)M_{\text{ASM}}(n, x, y, 1)(I - S^T)\]

where \(I_{ij} = \delta_{i,j}\) and \(S_{ij} = \delta_{i,j+1}\).

Therefore,

\[Z_{\text{DPP}}(n, x, y, 1) = Z_{\text{ASM}}(n, x, y, 1)\]
We have

\[(I - S)M_{\text{DPP}}(n, x, y, 1)(I + (\omega - 1)S^T) = (I + (x - \omega y - 1)S)M_{\text{ASM}}(n, x, y, 1)(I - S^T)\]

where \(I_{ij} = \delta_{i,j}\) and \(S_{ij} = \delta_{i,j+1}\).

Therefore,

\[Z_{\text{DPP}}(n, x, y, 1) = Z_{\text{ASM}}(n, x, y, 1)\]
We are working on various generalizations:

- At least one more statistic can be introduced: the **double refinement**. For ASMs this consists in recording the positions of the 1’s on both the first row and last row.

- There are **symmetry operations** on ASMs and DPPs. For example, there is an operation * which for ASMs is symmetry wrt a vertical axis, and for DPPs viewed as rhombus tilings is reflection in any of the three lines bisecting the central triangular hole.

De Gier, Pyatov and Zinn-Justin have proved in ’09 a conjecture of Mills, Robbins and Rumsey concerning these. The proof can probably be simplified and the result generalized.

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