• EULERIAN DIGRAPHS & TORIC CALABI-YAU VARIETIES •

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based on 1011.2963 [hep-th] and work in progress

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INTRODUCTION & MOTIVATION

Difficult to overstate the importance of toric Calabi-Yau geometry in modern theoretical physics.

Fundamental aspects of string theory like dualities and singularity resolution understood very concretely in such backgrounds.

Set of ground states at non-trivial superconformal IR fixed points of many supersymmetric gauge theories in four dimensions describe the coordinate ring of affine toric Calabi-Yau varieties.

Best understood setup is for D3-branes in IIB string theory probing a toric conical singularity – near the singularity, transverse space is an affine toric Calabi-Yau three-fold.

Singularity data encodes both superpotential and gauge-matter couplings in holographically dual superconformal field theory in terms of a quiver representation of the gauge symmetry group.

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Details of this branch often the key to unlocking more complicated phase structure and understanding holography
– systematic analyses by Hanany et al via forward algorithm, dimer models and brane tilings.

Vanishing first Chern class ⇔ cancellation of gauge anomalies at one-loop ⇔ quiver representation based on directed graph (digraph) with all vertices balanced.

Whence, for connected quivers, digraph must be eulerian.

But

- Not all eulerian digraphs compatible with toric superpotential – admissible ones encoded by brane tilings.
- Seiberg duality relates different admissible quivers which give same vacuum moduli space.

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Convenient physical description of affine toric Calabi-Yau varieties in terms of a superconformal gauged linear sigma model (GLSM).

Data from dimensional reduction of a supersymmetric theory in four dimensions with an abelian gauge group, n gauge superfields (labelled $i = 1, ..., n$) and e chiral matter superfields (labelled $a = 1, ..., e$) with integer charges Q_{ia}.

In addition, one must choose constants t_i for the Fayet-Iliopoulos (FI) parameters and a gauge-invariant, holomorphic function $\mathfrak{W}$ of the matter fields X_a defining the superpotential.

The Higgs branch of the vacuum moduli space contains the gauge-inequivalent constant matter fields which solve the D-term equations $\sum_{a=1}^{e} Q_{ia} |X_a|^2 = t_i$ – defines a Kähler quotient of $\mathbb{C}^e$.

If all $t_i = 0$, this branch contains a conical singularity at $X_a = 0$.

Non-anomalous superconformal symmetry requires $\sum_{a=1}^{e} Q_{ia} = 0$, ensuring this branch has vanishing first Chern class.

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Can define a superconformal GLSM by encoding the matter field charges by a quiver representation based on any eulerian digraph with n vertices and e arrows.

Our aim is examine the structure of a particular class of affine toric Calabi-Yau varieties which can be thought of physically as Higgs branches in superconformal GLSMs based on eulerian digraphs (with all FI parameters set to zero).

Why?

Can take advantage of some structure theory for eulerian digraphs to understand the associated Calabi-Yau geometries in more detail.

How?

Generate eulerian digraphs by iterating elementary graph-theoretic moves and derive their effect on the convex polytopes which encode the associated toric Calabi-Yau varieties.

Beware!

This is not the same as the auxiliary GLSM for the vacuum moduli space of an abelian quiver gauge theory based on a brane tiling.

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DIRECTED GRAPHS

Digraph $\vec{G}$ consists of a set of vertices V and a set of arrows A, with each $a \in A$ assigned $(v, w) \in V \times V$ (if (v, v) then a is a loop).

i.e. it is a graph equipped with an orientation.

Take V and A finite with $|V| = n$ and $|A| = e$ and define $t := e - n$.

Arrow a is simple if no other arrow in A is assigned the same (v, w) (or undirected simple if it is the only arrow connecting v and w).

Number of arrow heads/tails in $\vec{G}$ touching vertex $v \in V$ is called in-/out-degree $\deg^{\mp}(v)$.

Handshaking lemma: $\sum_{v \in V} \deg^{+}(v) = \sum_{v \in V} \deg^{-}(v) = e$.

$\vec{G}$ is balanced if $\deg^{+}(v) = \deg^{-}(v)$ of all $v \in V$.

Balanced $\vec{G}$ called k-regular if $\deg^{+}(v) = k$ for all $v \in V$, so $kn = e$.

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A walk in $\vec{G}$ is a sequence $(i_1 \xrightarrow{a_1} i_2 \xrightarrow{a_2} i_3 \ldots)$ where successive vertices $(i_p, i_{p+1}) \in V \times V$ are assigned to an arrow $a_p \in A$.

A path (cycle) is a (closed) walk with no repeated vertices.

A trail (circuit) is a (closed) walk with no repeated arrows.

$\vec{G}$ is strongly connected if $\exists$ a path between any pair of vertices in V (or weakly connected if $\exists$ an undirected path between any pair of vertices in V).

Path (cycle) is hamiltonian if it contains each vertex in V once – $\vec{G}$ is hamiltonian if it admits a hamiltonian cycle.

Trail (circuit) is eulerian if it traverses each arrow in A once – $\vec{G}$ is eulerian if it admits an eulerian circuit.

Characterising hamiltonian digraphs is difficult but there is a straightforward characterisation of eulerian digraphs.

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Characterising hamiltonian digraphs is difficult but there is a straightforward characterisation of eulerian digraphs.

A walk in $\vec{G}$ is a sequence $(i_1 \xrightarrow{a_1} i_2 \xrightarrow{a_2} i_3 \ldots)$ where successive vertices $(i_p, i_{p+1}) \in V \times V$ are assigned to an arrow $a_p \in A$.

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It is provided by the equivalent statements:

- $\vec{G}$ is eulerian.
- $\vec{G}$ is weakly connected and balanced ($\Rightarrow$ it is strongly connected).
- $\vec{G}$ is strongly connected and A can be partitioned into cycle digraphs on subsets of V.

Let $\mathfrak{G}$ denote the set of all eulerian digraphs and $\mathfrak{G}_k \subset \mathfrak{G}$ the subset of k-regular elements.

Any eulerian circuit in $\vec{G} \in \mathfrak{G}$ can be represented by a sequence $(i_1 i_2 \ldots i_e)$ of vertices around $\vec{C}_e$ labelled such that each $i_a \in \{1, \ldots, n\}$ with precisely $t = e - n$ labels repeated.

If $\vec{G} \in \mathfrak{G}_k$ then $t = (k-1)\,n$ and each vertex must appear exactly k times in any eulerian circuit

– if $\vec{G} \in \mathfrak{G}_1$ then it is isomorphic to $\vec{C}_n$.
– if $\vec{G} \in \mathfrak{G}_2$ then view an eulerian circuit as a chord diagram in $\vec{C}_{2n}$ with n chords connecting pairs of identical vertices.

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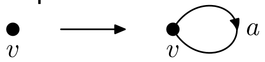
GENERATING EULERIAN DIGRAPHS

- Move I: Addition of a loop.



$e \to e+1$, $t \to t+1$ and $\deg^+(v) \to \deg^+(v)+1$.

- Move II: Subdivision of an arrow (or loop).

$e \to e+1$, $n \to n+1$ and $\deg^+(x) = 1$. Never creates a loop.

Reverse move called smoothing and $\vec{G}$ is smooth if it contains no vertices with out-degree one.

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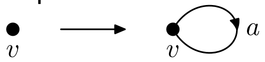

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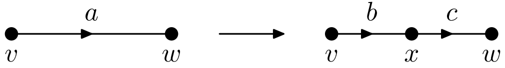

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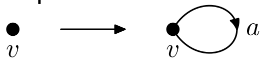

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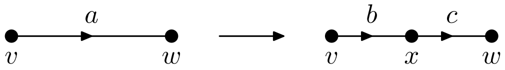

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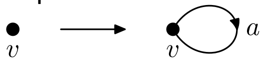

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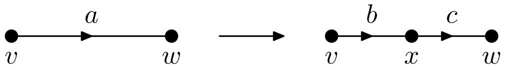

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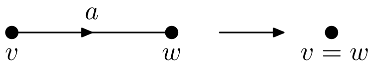

$e \to e-1$, $n \to n-1$ and $\deg^+(v) + \deg^+(w) - 1 = \deg^+(v = w)$.
Never creates a loop or subdivision. (It is written $\vec{G} \to \vec{G}/a$.)

- Move IV: Simple immersion of a pair of arrows.

$e \to e+2$, $n \to n+1$, $t \to t+1$ and $\deg^+(v) = 2$.

Reverse move called splitting an out-degree two vertex, which can be done in two ways
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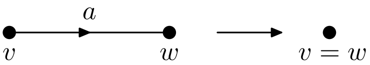

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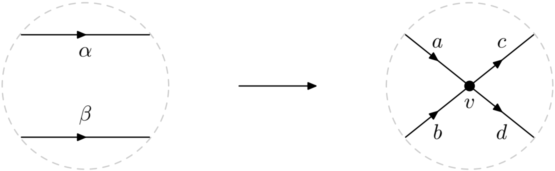

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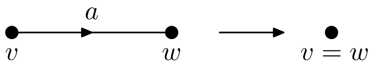

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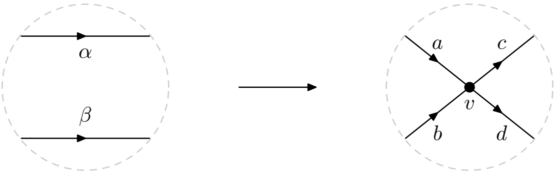

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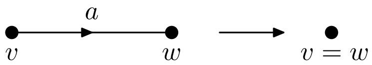

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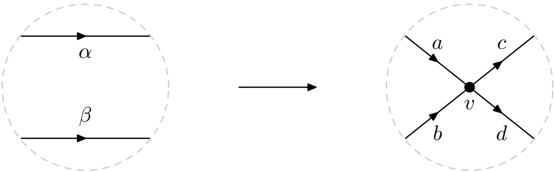

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A rough sketch of the construction is as follows:

- Moves I and II generate $\mathfrak{G}$ from $\mathfrak{F}$ (and the trivial graph).

$\vec{G}$ is smooth $\Leftrightarrow$ $\deg^+(v) > 1$ for all $v \in V$ and write handshaking lemma as $\sum_{v\in V} k(v) = t$, where each $k(v) := \deg^+(v) - 1 > 0$

$\Rightarrow$ $\vec{G}$ has $e \geq 2n$, with $e = 2n$ only if $\vec{G}$ is 2-regular.

- For fixed t, members of family $\mathfrak{F}^{[t]} \subset \mathfrak{F}$ have $2 \leq n \leq t$ vertices

– only parents in $\mathfrak{F}_2^{[t]}$ have $n = t$.

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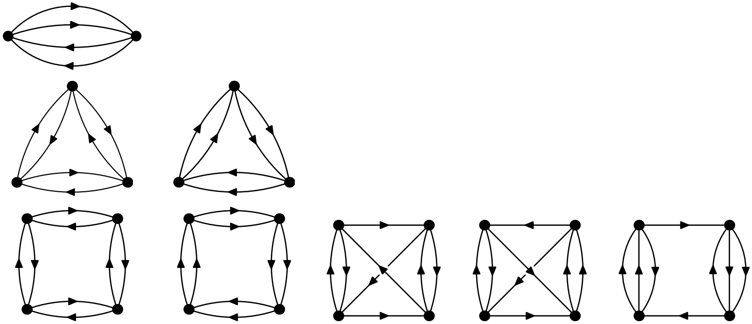
Some examples:



Elements in $\mathfrak{F}_2^{[t]}$ are drawn in row $t-1$ for $t=2,3,4$.

TORIC GEOMETRY FROM QUIVERS

Label vertices $i = 1, ..., n$ and arrows $a = 1, ..., e$ in $\vec{G}$ to fix a basis for the quiver representation of $\mathcal{G} \cong U(1)^n$ acting on $V \cong \mathbb{C}^e$ via

$$\begin{aligned} \mathcal{G} \times V &\to V \\ ((e^{\sqrt{-1}\theta_i}), (X_a)) &\mapsto \left(e^{\sqrt{-1}\sum_{i=1}^n \theta_i Q_{ia}} X_a\right) \end{aligned}$$

in terms of an incidence matrix with each component Q_{ia} equal to ± 1 if arrow a points to/from vertex i or zero otherwise.

$\sum_{a=1}^e Q_{ia} = 0$ whenever $\vec{G} \in \mathfrak{G}^{[t]}$.

Every arrow has one head and one tail so $\sum_{i=1}^n Q_{ia} \equiv 0$ ensuring quiver representation is not faithful

– kernel $\mathcal{K}$ contains diagonal $U(1) < U(1)^n$ for any loopless and weakly connected $\vec{G}$, leading to effective action of $\mathcal{H} = \mathcal{G}/\mathcal{K}$ on V.

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Toric geometry of $\mathcal{M}_{\vec{G}}$ encoded by convex rational polyhedral cone

$$\Lambda_{\vec{G}} = \text{Cone}(\Psi_{\vec{G}}) = \left\{ \sum_{a=1}^{e} \zeta_a \, \boldsymbol{\nu}_a \, \middle| \, \forall \, \zeta_a \in \mathbb{R}_{\geq 0} \right\} \subset \mathbb{R}^{t+1}$$

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$\Psi_{\vec{G}}$ is minimal rational generating set for $\Lambda_{\vec{G}}$, with all $\boldsymbol{\nu}_a$ primitive.

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$\vec{G}$ is a loopless eulerian digraph

⇒ $\mathcal{M}_{\vec{G}}$ is an affine toric Calai-Yau variety ($c_1(\mathcal{M}_{\vec{G}}) = 0$ only if $\sum_{a=1}^{e} Q_{ia} = 0$).

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- Leaves $SL(t, \mathbb{Z}) < SL(t+1, \mathbb{Z})$ unfixed.
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GENERATING TORIC CALABI-YAU VARIETIES

For any $\vec{G} \in \mathfrak{G}^{[t]}$, what do moves I-IV do to $\Delta_{\vec{G}} \subset \mathbb{R}^t$ encoding $\mathcal{M}_{\vec{G}}$?

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- Move II: Does not modify $\Delta_{\vec{G}}$ leaving $\mathcal{M}_{\vec{G}}$ invariant.

(cf. 'edge-doubling' in a brane tiling.)

- Move III: $\Delta_{\vec{G}} \to \Delta_{\vec{G}/a} = \text{Conv}(\psi_{\vec{G}} \backslash v_a) \subset \mathbb{R}^t$ and $\mathcal{M}_{\vec{G}} \to \mathcal{M}_{\vec{G}/a}$ involving quotient of $\mathbb{C}^e \backslash \mathbb{C}^*_a$ by $\mathcal{H}_{\mathbb{C}}/\mathbb{C}^*_{vw}$.

– natural physical interpretation via Higgsing matter field X_a in superconformal field theory which breaks $U(1)_{vw}$ gauge subgroup.

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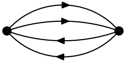
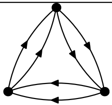
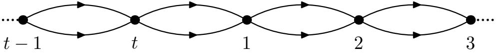
	$\mathcal{C}(T^{1,1})$
	$\mathcal{C}(Q^{1,1,1})$
	$\mathcal{C}(SU(2)^t/U(1)^{t-1})$

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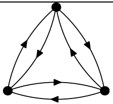
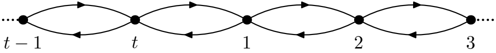
	$[0,1]*[0,1]*[0,1] = \triangle * \triangle$
	$\sigma_{t-1} * \sigma_{t-1}$

OPEN QUESTIONS

Apply to more interesting superconformal quiver gauge theories – need to incorporate a superpotential in the construction.

Interesting to consider brane tilings. Data $\tau_{\vec{G}}$ is a bipartite tiling of $\mathbf{T}^2$ with n faces, e edges and $t = e - n$ vertices – encodes both $\vec{G} \in \mathfrak{G}^{[t]}$ and a toric superpotential.

- Function mapping $\tau_{\vec{G}} \mapsto \vec{G}$ not bijective.
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Characterise composite moves which generate brane tilings encoding superconformal quiver gauge theories and effect of these moves on their vacuum moduli spaces? Watch this space...

Parent construction of M2-brane moduli spaces from D3-brane moduli spaces (via certain quotient involving Chern–Simons levels) – implications for toric duality or exact superconformal symmetry via ‘F-maximisation’?

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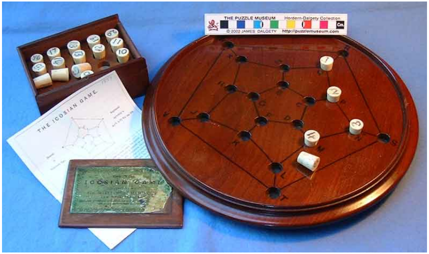
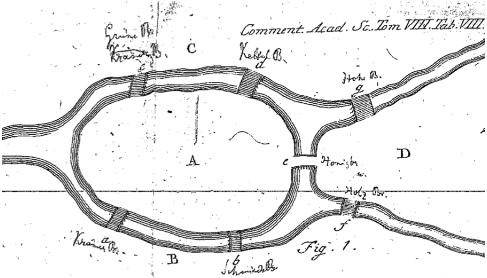
'ROUTES TO IMPACT'?



Hamilton's Icosian Game (1857)

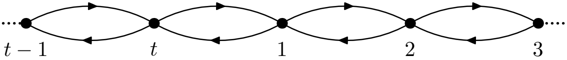
– "too easy, even for children!" ✗



Bridges of Königsberg (1735)

– "it is impossible!" [Euler] ✗

EXAMPLES



Move IV on arrows α, β connecting vertices t and 1 in $\vec{A}_t$ gives $\vec{A}_{t+1}$.

$(...(t-1)t\alpha 1\beta t(t-1)...212...) \to (...(t-1)tavc1bvdt(t-1)...212...)$

– vertices 2, 3,..., t, v all interlaced only with $1 \Rightarrow$ can take all $\gamma = \gamma^\circ$ and $\Delta_{\vec{A}_{t+1}} = \Delta_{\vec{A}_t} * [0,1] \subset \mathbb{R}^{t+1}$ defined recursively with

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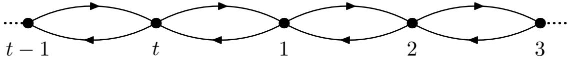

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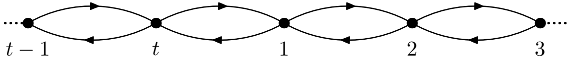

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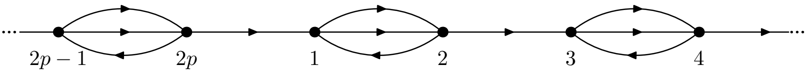


First perform move II on arrow connecting vertices $2p$ and 1 in $\vec{O}_p$ then move I on new vertex w.

Take arrows α, β to be new loop and arrow pointing from w to 1 then perform move IV to give $\vec{O}_{p+1}$.

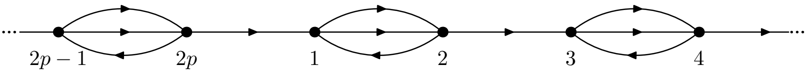
$(12123434...(2p-1)(2p)(2p-1)(2p))$

$\xrightarrow{\text{II+I}} (12123434...(2p-1)(2p)(2p-1)(2p)w\alpha w\beta)$

$\xrightarrow{\text{IV}} (12123434...(2p-1)(2p)(2p-1)(2p)wavcwbvd)$

– only vertex pairs $2i-1, 2i$ ($i = 1, ..., p$) and w, v are interlaced $\Rightarrow$ take all $\gamma = \gamma^\circ$ and $\Delta_{\vec{O}_{p+1}} = \Pi(\Delta_{\vec{O}_p}) * [0,1] \subset \mathbb{R}^{2(p+1)}$ with

$\Delta_{\vec{O}_p} \subset \mathbb{R}^{2p}$ convex hull of corners of unit squares in p planes $\mathbb{R}^2_i \subset \mathbb{R}^{2p}$ with $\mathbb{R}^{2p} = \cup_{i=1}^p \mathbb{R}^2_i$ and $\cap_{i=1}^p \mathbb{R}^2_i = \mathbf{0}$.



First perform move II on arrow connecting vertices $2p$ and 1 in $\vec{O}_p$ then move I on new vertex w.

Take arrows α, β to be new loop and arrow pointing from w to 1 then perform move IV to give $\vec{O}_{p+1}$.

$(12123434...(2p-1)(2p)(2p-1)(2p))$

$\xrightarrow{\text{II+I}} (12123434...(2p-1)(2p)(2p-1)(2p)w\alpha w\beta)$

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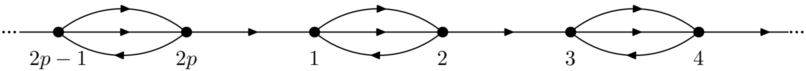

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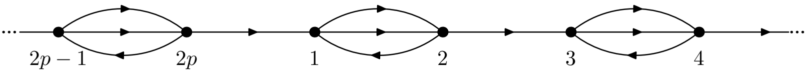

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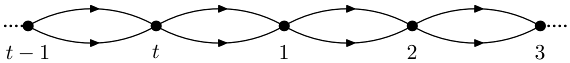


Move IV on arrows α, β connecting vertices t and 1 in $\vec{B}_t$ gives $\vec{B}_{t+1}$.

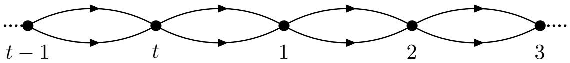
$(...(t-1)t\alpha 12...(t-1)t\beta 12...) \to (...(t-1)tavc12...(t-1)tbvd12...)$

– every vertex pair is interlaced (interlace graph of $\vec{B}_t$ is K_t).

Label i, $t+i$ arrow pairs pointing from vertex i to $i+1$ in $\vec{B}_t$ then integral vectors in $\psi_{\vec{B}_t}$ obey $\boldsymbol{v}_i + \boldsymbol{v}_{t+i} = (1, ..., 1) \in \mathbb{Z}^t$ (they end on opposite corners of unit hypercube $[0,1]^t \subset \mathbb{R}^t$).

Representative $\Delta_{\vec{B}_t} \subset \mathbb{R}^t$ defined by $\boldsymbol{v}_1 = \boldsymbol{e}_0$, $\boldsymbol{v}_i = \sum_{j=2}^{i} \boldsymbol{e}_j$ ($i = 2, ..., t$), where $\{\boldsymbol{e}_0, ..., \boldsymbol{e}_t\}$ are vertices of unit simplex $\sigma_t \subset \mathbb{R}^t$.

$\mathcal{M}_{\vec{B}_t}$ real metric cone over compact homogeneous Sasaki-Einstein manifold $SU(2)^t/U(1)^{t-1}$ (e.g. $T^{1,1}$ for $t=2$, $Q^{1,1,1}$ for $t=3$).



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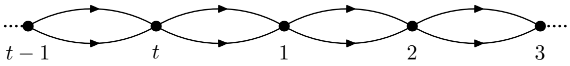

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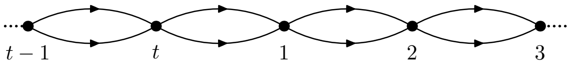

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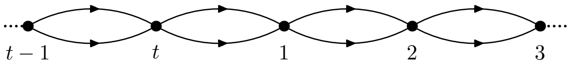

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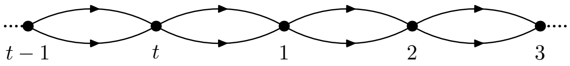

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