Non-chiral current algebras for deformed supergroup WZW models

Anatoly Konechny
Heriot-Watt University

September 21, 2011, EMPG seminar, Edinburgh

Based on a joint work with Thomas Quella (JHEP03(2011)124)

Outline

- Motivation
- Results obtained using perturbation theory
- Non-perturbative reformulation of the model
- Discussion

Motivation and prior work

Conformally invariant 2D sigma-models whose target space is a superspace emerge in

- disordered systems of condensed matter theory
- covariant quantization of superstrings on $AdS_d \times S^d$ backgrounds

The string background with pure Neveu-Schwarz flux can be formulated in terms of a WZW model on the supergroup $\mathrm{PSU}(1,1|2)$ (Berkovits, Vafa, Witten, 1999). In such a formulation one can describe deformations corresponding to switching on a mixture of N-S and R-R fluxes which preserves the full isometry of $G = \mathrm{PSU}(1,1|2)$ which is isomorphic to $G \times G$. In the operator description this is described by perturbing the WZW model by an operator

$$:J^a \phi_{ab} \bar{J}^b:(z,\bar{z})$$

where ϕ_{ab} is the primary field corresponding to the adjoint representation.

Geometrically this deformation is described as a principle chiral model on G with a Wess-Zumino term. It was first argued in Bershadsky, Zhukov, Vaintrob (1999) that such principal chiral models are conformal (for any value of the WZ term) when the supergroup has a vanishing Killing form. This happens for $\mathrm{PSU}(1,1|2)$ and for its generalizations $\mathrm{PSL(n|n)}$. Their arguments were generalized to general supergoups with vanishing Killing form and to their cosets:Kagan, Young (2005); Babichenko(2006) One can think of supermatrices from $\mathrm{SL(n|n)}$ as of block $2n \times 2n$ matrices

$$M = \begin{pmatrix} A & | & B \\ -- & -- & -- \\ C & | & D \end{pmatrix}$$

with the condition $\mathrm{Sdet}(M) = \det(A)/\det(B) = 1$.

Projecting out the multiples of identity one obtains the supergroup $\mathrm{PSL}(n|n)$. The supergroup $\mathrm{PSU}(1,1|2)$ is a real form or $\mathrm{PSL}(2|2)$. Another interesting deformation of the WZW theories on these supergroups is an isotropic current-current deformation:

$$\mathrm{Str}(J\bar{J}) = \kappa_{ba} :J^a \bar{J}^b: (z, \bar{z})$$

This deformation only preserves the diagonal subroup of the isometry supergroup $G \subset G \times G$.
Both deformations are exactly marginal due to the vanishing Killing form. The conserved currents corresponding to the global symmetries in the deformed models do not split into holomorphic and antiholomorphic components. The conservation equations are

$$\partial \bar{J}^a(z, \bar{z}) + \bar{\partial} J^a(z, \bar{z}) = 0 \,.$$

The above is usually the case in massive theories with conserved currents. In CFT's this only happens when unitarity is violated. Suppose ϕ is a CFT primary of weights $(n, 0)$ then

$$||\bar{L}_{-1}|\phi\rangle||^2 = \langle\phi|\bar{L}_1\bar{L}_{-1}|\phi\rangle = \langle\phi|2\bar{L}_0|\phi\rangle = 0\,.$$

An OPE algebra for conserved currents in massive theories were investigated in M.Luescher (1978); D. Bernard (1991) . The constraints imposed by current conservation were explored and an infinite tower of non-locally conserved charges were constracted. The general constraints on OPE were more recently generalized to parity-nonpreserving theories by S.Ashok, R. Benichou and J. Troost (2009). The current algebra in the $G \times G$-preserving deformation described above was studied in S.Ashok, R. Benichou and J. Troost (2009), (2009); R. Benichou and J. Troost (2010); R. Benichou (2011). An infinite set of conserved currents were put forward for this deformation. Those conserved quantities stem from the quantum Maurer-Cartan equation. Our work was much inspired by those papers.

Many of the results obtained for the $G \times G$-preserving deformation depend on certain assumptions involving the adjoint primary operator$\phi_{ab}(z, \bar{z})$ whose properties at present are not under analytic control. We study the second deformation where no such problem arises. We focus on the algebra generated by the deformed currents which is of interest at least for three reasons

- one would hope that as in the massive case studied by D. Bernard there is an infinite tower of conserved non-local charges constructed out of the currents. The interplay between conformal symmetry and the integrable structure (Yangian) would be interesting to explore
- one could hope that the non-chiral current algebra may be useful for organizing the spectrum
- it would be a new OPE algebra generalizing vertex operator algebras in unitary CFT's

For the current-current deformation we found that despite the absence of geometric reasons (no Maurer-Cartan equation) the quantum equation of motion takes the same form as in D. Bernard (1991) which makes possible the construction of Yangian charges.

The model

The WZW model on a supergroup G has local currents $J^a(z)$ and $\bar{J}^b(\bar{z})$ obeying the OPE

$$J^a(z)J^b(w) = \frac{k\kappa^{ab}}{(z-w)^2} + \frac{if^{ab}{}_c J^c(w)}{z-w} + \ldots$$

$$\bar{J}^a(\bar{z})\bar{J}^b(\bar{w}) = \frac{k\kappa^{ab}}{(\bar{z}-\bar{w})^2} + \frac{if^{ab}{}_c \bar{J}^c(\bar{w})}{\bar{z}-\bar{w}} + \ldots$$

where κ^{ab} is the non-degenerate invariant 2-form and $f^{ab}{}_c$ are the structure constants. They satisfy

$$\kappa^{ab} = (-1)^a \kappa^{ba}\,, \quad f^{ba}{}_c = -(-1)^{ab} f^{ab}{}_c$$

$$f^{ab}{}_d f^{dc}{}_e + (-1)^{c(a+b)} f^{ca}{}_d f^{db}{}_e + (-1)^{a(b+c)} f^{bc}{}_d f^{da}{}_e = 0$$

- the graded Jacobi identity. The vanishing of the Killing form implies that

$$f^{acd} f^b{}_{dc} = 0$$

Exact two-point functions of currents

In the deformed theory the operators representing the deformed currents are the bare WZW currents inserted into perturbation theory series. The dimension and spin are conserved and there are no fields with which the currents could form a Joardan block. Formally the deformed two-point functions are

$$\langle J^a(z_1,\bar{z}_1)J^b(z_2,\bar{z}_2)\rangle_\lambda = \langle J^a(z_1)J^b(z_2)\exp\left(-\frac{\lambda}{k\pi}\int d^2w\ :J^e\bar{J}^r:\ \kappa_{re}\right)\rangle_0$$

Two remarks are in order.Firstly in logarithmic theories typically

$$\langle \mathbf{1}\rangle = 0$$

This is not the case for some free field realizations of WZW models; otherwise one should treat such correlators more formally as a means to compute OPE's.

Secondly, the perturbation series integrals have divergences. The integrands (for finite separation) are proportional to

$$\langle J^a(z_1)J^b(z_2)J^{e_1}(w_1)\dots J^{e_n}(w_n)\rangle_0\langle \bar{J}^{e_1}(\bar{w}_1)\dots J^{e_n}(\bar{w}_n)\rangle_0$$

Each correlator is given by the sum of singular terms in the OPE contractions and thus looks like a rational function times an invariant tensor constructed from κ^{ab} and f^{abc}.

Fixing the method of subtraction is equivalent to defining the composite operator $:J^e\bar{J}^r:\kappa_{re}$, i.e. we define its correlation functions as distributions. Contact term ambiguities in these correlators are related by coupling constant redefinitions. There are no infrared divergences.

Noting that f^{abc} is the only invariant 3-tensor for the supergroups of interest, and that one cannot form any invariant scalar quantities from f^{abc} due to the vanishing of C_{ad}, one only needs to sum the abelian part of the perturbation series above. A particularly nice prescription for perturbative integrals in the abelian theory were worked out by G. Moore (1993).For tori the corresponding couplings coincide with certain canonical coordinates on the moduli space, such as Klein coordinates for T^2. We adopt Moore's coordinate λ. Summing up the perturbation series we obtain

$$\langle J^a(z_1,\bar{z}_1)J^b(z_2,\bar{z}_2)\rangle_\lambda = \frac{k\kappa^{ab}}{(1-\lambda^2)z_{12}^2}\,,$$

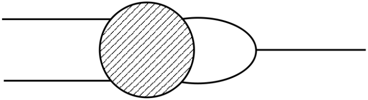
$$\langle \bar{J}^a(z_1,\bar{z}_1)\bar{J}^b(z_2,\bar{z}_2)\rangle_\lambda = \frac{k\kappa^{ab}}{(1-\lambda^2)\bar{z}_{12}^2}\,, \quad \langle J^a(z_1,\bar{z}_1)\bar{J}^b(z_2,\bar{z}_2)\rangle_\lambda = 0\,.$$

Exact three-point functions of currents

It was argued by Bershadsky, Zhukov and Vaintrob that to compute a deformed 3-point function it suffices to retain terms in perturbation theory containing only one factor of f^{abc}. That argumant works if we assume that there are only three traceless invariant four-tensors:

$$f_{ab}{}^{e} f_{cde}\,, \quad f_{ac}{}^{e} f_{bde}\,, \quad \kappa_{ab}\kappa_{cd} + (-1)^{bc}\kappa_{ac}\kappa_{bd} + \kappa_{ad}\kappa_{bc}$$

Any 3-tensor resulting from a contraction of more than two structure constants can be represented diagrammatically as



The blob containing four external lines must correspond to a traceless tensor as there are no corrections to the invariant metric κ^{ab}Each of the above three traceless tensors vanishes when contracted with the structure constants on any two indices. Extracting factors of f^{abc} via the singularities in the OPE's and summing the remaining abelian perturbation series using Moore's prescription we obtain

$$\langle J^a(z_1,\bar{z}_1)J^b(z_2,\bar{z}_2)J^c(z_3,\bar{z}_3)\rangle_\lambda = \frac{1-\lambda^3}{(1-\lambda^2)^3}\left[\frac{-ikf^{abc}}{z_{12}z_{23}z_{31}}\right]$$

$$\langle J^a(z_1,\bar{z}_1)J^b(z_2,\bar{z}_2)\bar{J}^c(z_3,\bar{z}_3)\rangle_\lambda = \frac{\lambda(1-\lambda)}{(1-\lambda^2)^3}\left[\frac{-ikf^{abc}\bar{z}_{12}}{z_{12}^2\bar{z}_{23}\bar{z}_{31}}\right]$$

Using the above methods we can only obtain $1/k$ expansion for four and higher n-point functions. Thus we computed

$$\langle J^a(z_1,\bar{z}_1)J^b(z_2,\bar{z}_2)J^c(z_3,\bar{z}_3)J^d(z_4,\bar{z}_4)\rangle_\lambda = k^2 R_1 + kR_2$$
$$+k\frac{\lambda^2(1-\lambda)^2}{(1-\lambda^2)^5}\Big[-f^{ab}{}_r f^{rcd}\frac{1}{z_{34}^2 z_{12}^2}\ln\left|\frac{z_{13}z_{24}}{z_{23}z_{14}}\right|^2$$
$$+(-1)^{a(b+c)}f^{ad}{}_r f^{bcr}\frac{1}{z_{14}^2 z_{23}^2}\ln\left|\frac{z_{24}z_{13}}{z_{12}z_{34}}\right|^2$$
$$-(-1)^{ab}f^{ac}{}_r f^{brd}\frac{1}{z_{13}^2 z_{24}^2}\ln\left|\frac{z_{23}z_{14}}{z_{12}z_{34}}\right|^2\Big] + \mathcal{O}(k^0)$$

where R_1 and R_2 are rational functions of z_{12}, z_{23}, z_{34} known explicitly and to all orders in λ.

OPE algebra of currents

For the deformed OPE's of currents we can take a basis of deformed operators labelled by operators in the vaccum sector of the WZW theory. Note that $:J^a\bar{J}^b:(z,\bar{z})$ stands for an operator in the *deformed* theory whose correlators are obtained by inserting the bare composite $:J^a\bar{J}^b:$ into the perturbation seris and taking integrals using Moore's prescription or its extension. Such an operator in general will not coincide with the normal ordered product of the deformed currents:

$$:J^a\bar{J}^b:(z,\bar{z}) \neq \lim_{:z_1\rightarrow z_2:} J^a(z_1,\bar{z}_1)\bar{J}^b(z_2,\bar{z}_2)\ .$$

To compute the leading order OPE coefficients we used the method of R. Guida and N. Magnoli (1996).

For the JJ OPE we can write the folowing ansatz

$$J^a(z_1,\bar{z}_1)\,J^b(z_2,\bar{z}_2) = \frac{k\kappa^{ab}(\lambda)}{(z_1-z_2)^2} + \frac{i f^{ab}{}_c(\lambda) J^c(z_2,\bar{z}_2)}{z_1-z_2}$$
$$+ g^{ab}_{cd}(\lambda) :J^c J^d: (z_2,\bar{z}_2) + h^{ab}_c(\lambda)\partial J^c(z_2,\bar{z}_2)$$
$$+ \frac{\bar{z}_1-\bar{z}_2}{z_1-z_2} t^{ab}_{cd}(\lambda)\ :J^c\bar{J}^d: (z_2,\bar{z}_2) + \frac{\bar{z}_1-\bar{z}_2}{(z_1-z_2)^2} u^{ab}_c(\lambda)\bar{J}^c(z_2,\bar{z}_2)$$
$$+ \frac{(\bar{z}_1-\bar{z}_2)^2}{(z_1-z_2)^2} v^{ab}_c(\lambda)\bar{\partial}\bar{J}^c(z_2,\bar{z}_2) + \frac{(\bar{z}_1-\bar{z}_2)^2}{(z_1-z_2)^2} w^{ab}_{cd}(\lambda) :\bar{J}^c\bar{J}^d: (z_2,\bar{z}_2)$$
$$+ \quad \text{higher dimension fields}$$

The z-dependence contains rational parts dictated by the spin and scaling dimensions, which however can be decorated by logarithms. This does not happen for some of the leading singularities which are fixed by the exact three-point functions. Thus we have

$$\kappa^{ab}(\lambda) = \frac{\kappa^{ab}}{1-\lambda^2}\,, \quad f^{ab}{}_c(\lambda) = \frac{1-\lambda^3}{(1-\lambda^2)^2} f^{ab}{}_c\,, \quad u^{ab}_c(\lambda) = i f^{ab}{}_c \frac{\lambda(1-\lambda)}{(1-\lambda^2)^2}$$

The rest of the terms in the above ansatz we evaluated to the first order in λ. We found that operators$(-1)^{bd} f^a{}_{dg} f^{gb}{}_c :J^c \bar{J}^d:$ and $f^{ab}{}_c \bar{\partial} \bar{J}^c$ also appear on the RHS.
Similarly for the $J\bar{J}$ OPE we find

$$J^a(z_1,\bar{z}_1)\,\bar{J}^b(z_2,\bar{z}_2) = \frac{B^{ab}_c(\lambda)}{z_1-z_2}\,\bar{J}^c(z_2,\bar{z}_2) + \frac{C^{ab}_c(\lambda)}{\bar{z}_1-\bar{z}_2}\,J^c(z_2,\bar{z}_2)$$
$$-\frac{z_1-z_2}{\bar{z}_1-\bar{z}_2}\,i\lambda f^{ab}{}_c\,\partial J^c(z_2,\bar{z}_2)$$
$$-(-1)^{fs}\,\frac{\lambda}{k}\kappa_{ef} f^{ae}{}_c f^{bf}{}_d \ln\frac{|z_1-z_2|^2}{\epsilon^2}\;:J^c\bar{J}^d:(z_2,\bar{z}_2) + \mathcal{O}(\lambda^2)\,.$$

where

$$C^{ab}_c(\lambda) = B^{ab}_c(\lambda) = i f^{ab}{}_c \frac{\lambda(1-\lambda)}{(1-\lambda^2)^2}$$

are known to all orders in λ.

Equal time commutators

The equal time commutator algebra of the currents can be obtained from the most singular terms in the OPEs via the Bjorken-Johnson-Low limit:

$$\left[J^a_\mu(\sigma_1,0), J^b_\nu(\sigma_2,0)\right] = \lim_{\epsilon\to 0}\Big(J^a_\mu(\sigma_1,i\epsilon)J^b_\nu(\sigma_2,0) - J^b_\nu(\sigma_2,i\epsilon)J^a_\mu(\sigma_1,0)\Big)$$

We further compactify the spacial direction on a circle: $\sigma \sim \sigma + 2\pi$ and introduce the Fourier modes

$$J^a(\sigma,\tau) = i\sum_{n\in\mathbb{Z}} e^{-in\sigma} J_n(\tau)\,, \quad \bar{J}^a(\sigma,\tau) = -i\sum_{n\in\mathbb{Z}} e^{-in\sigma}\bar{J}_n(\tau)\,.$$

The answer looks most simple in terms of the modes

$$l_n^a(\tau) = J_n^a(\tau) - \lambda \bar{J}_n^a(\tau)\,, \quad r_n^a(\tau) = \bar{J}_n^a(\tau) - \lambda J_n^a(\tau)$$

We get two commuting copies of the affine current algebra:

$$\begin{aligned}
\left[l_n^a(\tau), l_m^b(\tau)\right] &= k\kappa^{ab} n\delta_{n,-m} + i f^{ab}{}_c l^c_{n+m}(\tau)\,, \\
\left[r_n^a(\tau), r_m^b(\tau)\right] &= -k\kappa^{ab} n\delta_{n,-m} + i f^{ab}{}_c r^c_{n+m}(\tau)\,, \\
\left[r_n^a(\tau), l_m^b(\tau)\right] &= 0\,.
\end{aligned}$$

Equations of motion

Based on dimension, spin conservation and global symmetry the deformed equations of motion must be of the form

$$\bar{\partial}^a(z,\bar{z}) = -\partial \bar{J}^a(z,\bar{z}) = iG(\lambda) f^a{}_{bc} :J^c \bar{J}^b: (z,\bar{z})$$

where we also used the fact that f^{abc} is the only invariant 3-tensor. Using the same diagrammatic reasoning as Bershadsky et al. we can show that

$$f^a{}_{bc} :J^c \bar{J}^b: (z,\bar{z}) = \lim_{w\to z} f^a{}_{bc} J^c(w,\bar{w}) \bar{J}^b(z,\bar{z})$$

where no singularities occur in taking the limit. Using this representation and the exact OPE coefficients of the currents we can find $G(\lambda)$ exactly by matching the leading singularities in the OPE with the currents on both sides of the equation of motion.

We obtain

$$\bar{\partial}^a(z,\bar{z}) = -\partial \bar{J}^a(z,\bar{z}) = -i\frac{\lambda}{(1+\lambda)k} f^a{}_{bc} :J^c \bar{J}^b: (z,\bar{z})$$

The form of this equation of motion is the same as the one considered by D. Bernard up to rescaling the currents. We thus expect our model to possess the same Yangian symmetries as in D. Bernard (1991).

Virasoro algebra

The Hamiltonian densities giving the equation of motion are

$$T(\sigma,\tau) = \left(\frac{1-\lambda^2}{2k}\right)\kappa_{dc} :J^c J^d: (\sigma,\tau)\,,$$

$$\bar{T}(\sigma,\tau) = \left(\frac{1-\lambda^2}{2k}\right)\kappa_{dc} :\bar{J}^c \bar{J}^d: (\sigma,\tau)$$

so that

$$\bar{\partial} J^a(z,\bar{z}) = \frac{i}{2\pi}\Big[\int d\sigma\, \bar{T}(\sigma,\tau), J^a(z,\bar{z})\Big]\,,$$

$$\partial \bar{J}^a(z,\bar{z}) = \frac{i}{2\pi}\Big[\int d\sigma\, T(\sigma,\tau), \bar{J}^a(z,\bar{z})\Big]\,.$$

We found an argument demonstrating that both generators remain holomorphic (anti-holomorphic respectively) to all orders in λ.

Open questions/Future directions

- the perturbation theory methods can be applied to obtain the diagonal parts of the deformed conformal dimensions, we are currently trying to check those formulas agains the lattice data for the supersphere sigma models. (Work in progress with Candu, Quella, Schomerus)
- integrability and its interplay with conformal symnmetry needs to be thoroughly investigated
- string theory interpretation of the G-preserving deformation would be interesting to work out. Squashed AdS_3 with fluxes?
- it would be desirable to get better analytic control over the ϕ_{ab} primary to push further the results available for the $G \times G$-preserving deformation