Bounds for state degeneracies in two-dimensional conformal field theories

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Outline

Introduction: Landscape of 2d field theories

Conformal field theory (cft) shows up in many places
(thermodynamics, statistical mechanics, string world-sheet theories, duals to gravitational theories, etc.)

Part of reason:
Covariance under dilatation $\leftrightarrow$ fixed point for RG flow.

In 2d: Dilatation covariance $\rightarrow$ conformal covariance
is not so big a step [Polchinski '88] .
Hence: 2d cfts play important role in space of 2d field theories (recently [Douglas '10]).

Much effort into classifying cfts
[BPZ '84, FQS '85, Capelli-Itzykson-Zuber '87, Mathur-Mukhi-Sen '88 . . .] .
But we would also like to understand some gross physical features of cft classes.

Introduction: Landscape of 2d field theories

Recently: Limits in space of cfts explored. Question: How big can a gap between primary dimensions become?

- using holomorphic factorisation [Cardy '86, Höhn '07, Witten '07]
- using extended symmetry [Gaberdiel et al '08]
- in OPE [Rattazzi et al '08]
- dimension of lowest nontrivial primary [Hellerman '09]

Result: Gaps are below some universal bound, depending on input parameters.
Reason: Modular invariance (in unitary theories).

We tried to address another quite general question about cfts:

Our question

Using modular invariance, may there also be a general bound for the number of operators at a certain energy? For the entropy?

Objective

What could be a physical reason?

Holographic principle ['t Hooft '93, Susskind '95] $\longrightarrow$ number of marginal directions should not become infinitely large.

A mathematical one?

Maybe bound on Euler number of compact CY3s [Chang et al '01] $\longrightarrow$ could follow from bound on moduli space dimension of scft at $c_L = c_R = 9$.

Program

For a unitary cft at $\beta = 2\pi$, with some additional assumptions,

- derive an upper bound for the number of marginal operators
- derive (a lower and) an upper bound on the entropy

The setup

Consider a system periodic in two directions (torus of modulus τ) with

$$H = L_0 + \tilde{L}_0 - \frac{c_{tot}}{24}\,, \quad P = L_0 - \tilde{L}_0 - \frac{c_L - c_R}{24}$$

and partition function

$$Z[\tau] = \mathrm{tr}\exp\left(2\pi i\tau(L_0 - \frac{c_L}{24}) - 2\pi i\bar{\tau}(\tilde{L}_0 - \frac{c_R}{24})\right)\,.$$

Consider the generators of the modular group:

- $T : \tau \mapsto \tau + 1$ is simply the quantisation of P.
- $S \to -\frac{1}{\tau}$ has no interpretation in Hamiltonian formalism.
  Nevertheless it is a good quantum symmetry under quite broad conditions (locality of path integral).

We will only need S.

The setup

Restrict to rectangular torus with inverse temperature

$$T^{-1} = \beta = 2\pi\tau_2 \qquad (\text{if } \tau = \tau_1 + i\tau_2\,;\, \tau_1 = 0).$$

Assumption

- cft is unitary,
- H has discrete spectrum on finite volume.

$$Z[\beta] = \mathrm{tr}\exp(-\beta H) = \sum_n \exp(-\beta E_n)$$

E_n eigenvalues of H.

$S : \beta \mapsto \frac{4\pi^2}{\beta}$: self-dual temperature is $\beta = 2\pi$.

Exploiting modular invariance

S-transformation condition:

$$Z[\beta] = Z[\tfrac{4\pi^2}{\beta}] \tag{1}$$

Write $\beta = 2\pi\exp(s)$, then $S: s \mapsto -s$. Write $Z = Z(s)$.

Expand (1) for small s:

$$\begin{aligned} &Z(0) + s(\partial_s Z)(0) + \tfrac{1}{2}s^2(\partial_s^2 Z)(0) + \dots \\ &\qquad = Z(0) - s(\partial_s Z)(0) + \tfrac{1}{2}s^2(\partial_s^2 Z)(0) \pm \dots . \end{aligned}$$

With $\partial_s = \beta\partial_\beta$, this means

$$(\beta\partial_\beta)^p Z[\beta]\Big|_{\beta=2\pi} = 0 \quad \text{for odd } p\,.$$

Alternatively

$$\sum_n \exp(-2\pi E_n) f_p(E_n) = 0 \quad \text{for odd } p\,,$$

with

$$f_1(E) = -2\pi E\,, \quad f_3(E) = -(2\pi E)^3 + 3(2\pi E)^2 - (2\pi E)\,, \dots$$

Exploiting modular invariance

Linear combinations $\longrightarrow$ Modular invariance means, at $\beta = 2\pi$, that

S-transformation condition

$$\sum_n \exp(-2\pi E_n) f_F(E_n) = 0$$

$$\text{for } f_F(E) \equiv \exp(2\pi E)\, F(\beta\partial_\beta)\exp(-\beta E)\Big|_{\beta=2\pi},$$

for any odd function F.

Note that $f_p(E) = f_{F(x)=x^p}(E)$.

Lower bound for the entropy

Immediate consequence: A lower bound for the entropy density

$$\sigma = \log Z + \beta\langle E\rangle$$

at $\beta = 2\pi$. With $f_1(E) = -2\pi E$, we have

$$0 = \sum_n \exp(-2\pi E_n) f_1(E_n) = -2\pi\langle E\rangle\Big|_{\beta=2\pi} .$$

Since

$$\log Z \geq -2\pi E_0$$
$$\text{for } E_0 = -\tfrac{c_{tot}}{24} \text{ energy of ground state,}$$

we have

$$\sigma\Big|_{\beta=2\pi} \geq \frac{\pi c_{tot}}{12} .$$

Note:

This bound is exactly saturated in Kerr/cft [Guica et al '08, Castro et al '09] :
$\beta = 2\pi$, $c_L = c_R = 12J$, $S = 2\pi J$.

However, these formulae are asymptotic for large angular momentum J, and there are (most probably) corrections of order J^0.

Hence we do not claim that Kerr/cft is empty (or nonunitary) for finite J!

Elementary upper bound for lowest nontrivial dimension

In order to illustrate the method, let us derive an upper bound for the lowest nontrivial operator dimension in our unitary cft. [Hellerman '09]

Consider $0 = (\beta\partial_\beta)^p Z[\beta] = (\beta\partial_\beta)^p \sum_{n\geq 0} e^{-\beta E_n}$.

"Increasing p makes contribution from higher levels more important."

Consider $p = 1$, $p = 3$: Particular energy level n must have energy E_n

- high enough st. first derivative can vanish
- low enough st. third derivative also vanishes

Define "relative importance" of energy E by

$$I(E) := \left.\frac{(\beta\partial_\beta)^3 e^{-\beta E}}{\beta\partial_\beta e^{-\beta E}}\right|_{\beta=2\pi} = 4\pi^2 E^2 - 6\pi E + 1\,.$$

Notice: $I(E) = I(E_0)$ for $E = E_0$, but also for

$$E = E_+ \equiv \frac{3}{2\pi} - E_0\,.$$

In particular: $\begin{array}{ll} E_0 < E < E_+ & : I(E) < I(E_0), \\ E > E_+ & : I(E) > I(E_0). \end{array}$

Want to show: At least $I(E_1)$ must be smaller than $I(E_0)$. At $\beta = 2\pi$,

$$\beta\partial_\beta Z[\beta] = 0 \; : \quad \beta\partial_\beta e^{-\beta E_0} = -\sum_{n\geq 1} \beta\partial_\beta e^{-\beta E_n} \,,$$

$$(\beta\partial_\beta)^3 Z[\beta] = 0 \; : \quad (\beta\partial_\beta)^3 e^{-\beta E_0} = -\sum_{n\geq 1} (\beta\partial_\beta)^3 e^{-\beta E_n} \,.$$

$$I(E_0) = \frac{(\beta\partial_\beta)^3 e^{-\beta E_0}}{\beta\partial_\beta e^{-\beta E_0}} = \frac{\sum_{n\geq 1}(\beta\partial_\beta)^3 e^{-\beta E_n}}{\sum_{n\geq 1} \beta\partial_\beta e^{-\beta E_n}} = \frac{\sum_{n\geq 1} I(E_n) E_n e^{-\beta E_n}}{\sum_{n\geq 1} E_n e^{-\beta E_n}} \,.$$

Hence

$$0 \stackrel{!}{=} \frac{\sum_{n\geq 1}(I(E_n) - I(E_0)) E_n e^{-\beta E_n}}{\sum_{n\geq 1} E_n e^{-\beta E_n}} \,,$$

which is not possible for $I(E_1) > I(E_0)$. Therefore we must have

$$E_0 < E_1 < E_+ \,.$$

Note that this argument only tells us something if $c_{tot} < 24 - \frac{18}{\pi} \approx 18.270$.

What does this mean in terms of the S transformation condition as we presented it before?

We just used the old form

$$\sum_n \exp(-2\pi E_n) f_F(E_n) = 0 \qquad (2)$$

with the function f_F derived from

$$F(x) = x^3 - I(E_0)x = x(x^2 - 4\pi^2 E_0^2 + 6\pi E_0 - 1)\,,$$

i.e.

$$f_F(E) = f_3(E) - I(E_0) f_1(E) = -8\pi^3 E(E - E_0)(E - E_+)\,.$$

Notice that $f_F(E)$ vanishes at E_0, and is negative for $E > E_+$. If all E_n ($n \geq 1$) were to be E_+ or higher, (2) could not vanish.

General caveats to upper bounds of entropy

With our method, upper bounds for entropy or state degeneracies in a specific energy range will be difficult without further assumptions on the cft:

1. Control of unwanted contributions from states around vacuum:

- *Homogeneity*: Can take several copies of a modular invariant spectrum.
- *Continuum*: If $\rho(E) \geq 0$ is the state density, condition is $\int dE\, \rho(E) f_F(E) = 0$. Procedure as for upper bound of lowest dimension would fail if there is a continuum of states close to the vacuum:
  For a universal bound on state degeneracy in some energy range $E_0 < E_1 \leq E \leq E_2 \leq \infty$, can fix F st. f_F has opposite sign there and at vacuum. If family contains cfts with arbitrarily many states close to the vacuum, continuity of F may force us to admit arbitrarily many states in energy range (E_1, E_2).
  Strictly never have a continuum at vacuum, but there are sets of theories with asymptotically continuous spectrum (see again Kerr/cft).
- *Characters*: One could reorganise the terms in the partition function, *e.g.* into characters. Similar approach as before can separate vacuum representation from every other representation, but difference is small $\rightarrow$ seemed to give problems if we attempt to overcome continuum problem.

General caveats to upper bounds of entropy

2. Control of unwanted contributions from states at high energy:
   - *Large c*: As $E_0 \ll 0$, it becomes difficult to find $f_F(E)$ positive for $E = E_0$ and negative for all $E \geq E_0 + \Delta$, say.
     Reason: For $E_0 \to -\infty$, Δ/E_0 =const.,

     $$f_F(E) \equiv \exp(2\pi E)\, F(\beta\partial_\beta)\exp(-\beta E)\big|_{\beta=2\pi} \sim F(-2\pi E) \quad \text{for } E \sim E_0\,.$$

     F is odd, such that there will necessarily be positive contributions from far up in the spectrum ($E \gg 0$) for large c.

3. Control of unwanted contributions from states around marginality:
   - *Resolution*: Difficulty to fix an entropy associated to "marginality". While there may be an upper limit to the number of marginal states ($E = E_0 + 2$), there are classes of cfts where there are arbitrarily many states in $(E_0 + 2 - \epsilon, E_0 + 2 + \epsilon)$ (*e.g.* on tori of arbitrarily high radii).

Assumptions

Assumptions on our cft:

- *Unitarity* and *cluster decomposition*. With this we avoid the homogeneity problem.
- *Perturbative stability*, *i.e.* a spectrum without non-trivial relevant operators. With this assumption we avoid the continuum problem, and moreover will obtain results from the first-order derivative already.
- *Restricted central charge* $c_{tot} < 48$. Besides avoiding the large-c problem, this is also necessary to obtain a result from the first derivative.
- The resolutional problem will turn out not to exist in stable cfts with $c_{tot} < 48$.

Upper bound for the number of marginal operators

$N =$ number of primary operators at $E_0 + 2$.

Notice that in the following example we will not be able to distingush between scalars and $(h, 2-h)$ operators.
With our assumptions, the number of operators at $E_0 + 2$ is $N + 2$.

Let us now consider $F(x) = -\frac{x}{2\pi}$, $f_F(E) = E$. Condition reads

$$\begin{aligned} 0 = E_0 e^{-2\pi E_0} & \qquad \leftarrow \text{negative} \\ + (E_0+2)(N+2)e^{-2\pi(E_0+2)} & \qquad \leftarrow \text{positive if } c_{tot} < 48 \\ + \sum_{\Delta>2} (E_0+\Delta)e^{-2\pi(E_0+\Delta)} & \qquad \leftarrow \text{positive} \end{aligned}$$

Multiplication by $\exp(2\pi(E_0+2))$ leads to

$$\begin{aligned} 0 &< (E_0+2)(N+2) \\ &< (E_0+2)(N+2) + \sum_{\Delta>2}(E_0+\Delta)e^{-2\pi(\Delta-2)} = -E_0 e^{4\pi}\,, \end{aligned}$$

$$i.e. \qquad N < \frac{c_{tot}}{48 - c_{tot}} e^{4\pi} - 2\,.$$

Upper bound for the number of marginal operators

From bound on the lowest primary dimension, this bound is rigorously true for

$$24 - \frac{18}{\pi} < c_{tot} < 48\,.$$

For integer values of c in this range, the bound is

c_{tot}	$N^{\max}$	c_{tot}	$N^{\max}$	c_{tot}	$N^{\max}$
19	187’869	29	437’671	39	1’242’587
20	204’820	30	477’916	40	1’433’754
21	223’026	31	522’897	41	1’679’514
22	242’633	32	573’500	42	2’007’257
23	263’809	33	630’850	43	2’466’059
24	286’749	34	696’394	44	3’154’262
25	311’684	35	722’020	45	4’301’267
26	338’885	36	860’251	46	6’595’278
27	368’678	37	964’525	47	13’477’309
28	401’449	38	1’089’652	48	∞

Upper bound for the entropy

Starting from the same condition for $F(x) = x/2\pi$ again, we obtain

$$1 = \sum_{n \geq 1} \frac{E_n}{|E_0|} \exp(-2\pi(E_n + |E_0|)) \,. \qquad (3)$$

Our assumptions are such that

$$0 < E_1 \leq E_2 \leq \dots$$

The nth term in (3) is hence bounded below by

$$\frac{E_n}{|E_0|} \exp(-2\pi(E_n + |E_0|)) \geq \frac{E_1}{|E_0|} \exp(-2\pi(E_n + |E_0|))$$

Sum and multiply with $\frac{|E_0|}{E_1} \exp(2\pi|E_0|)$, add vacuum part $\exp(2\pi|E_0|)$:

$$Z[\beta = 2\pi] \leq (1 + \tfrac{|E_0|}{2-|E_0|}) \exp(2\pi|E_0|) \,,$$

which, with $\langle E \rangle \big|_{\beta=2\pi} = 0$, yields

$$\tfrac{\pi c_{tot}}{12} \leq \sigma\big|_{\beta=2\pi} \leq \tfrac{\pi c_{tot}}{12} + \log(\tfrac{48}{48-c_{tot}}) \,.$$

Further steps

What have we done so far?

- For a unitary cft at medium temperature with discrete spectrum, established upper bound for lowest nontrivial dimension.
- For a similar cft which is also stable and has $c_{tot} < 48$, established upper bounds for the number of marginal operators.
- Under the same assumptions, proven upper and lower bounds for the entropy.

Obviously there is much to do:

- Want more refined information. *E.g.*, bound number of scalar marginal operators, without spin operators. Way to achieve this: Use $(\tau\partial_\tau)^{p_1}(\bar{\tau}\partial_{\bar{\tau}})^{p_2} Z[\tau,\bar{\tau}]\big|_{\tau=i}$ for $p_1 + p_2$ odd.
- Weaker assumptions: No bound on central charge, no exclusion of relevant operators, no restriction of chiral algebra. Will have to deal with the problems mentioned. Maybe higher orders, or considering cft on surfaces of higher genus, will provide a clue.
- Want to apply this in string theory (cfts of CYs, Kerr/cft,. . .).