Rational matrix factorizations via defect functors

based on 1005.2117 and 1112.XXXX

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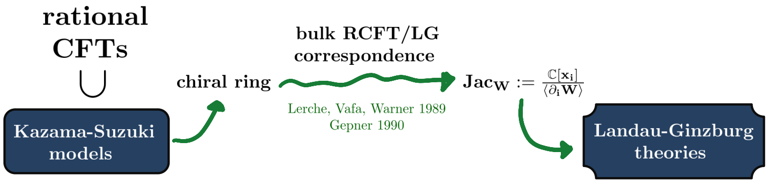
rational CFTs
Kazama-Suzuki models
chiral ring
bulk RCFT/LG correspondence
Lerche, Vafa, Warner 1989
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$\mathrm{Jac}_W := \frac{\mathbb{C}[x_i]}{\langle \partial_i W \rangle}$
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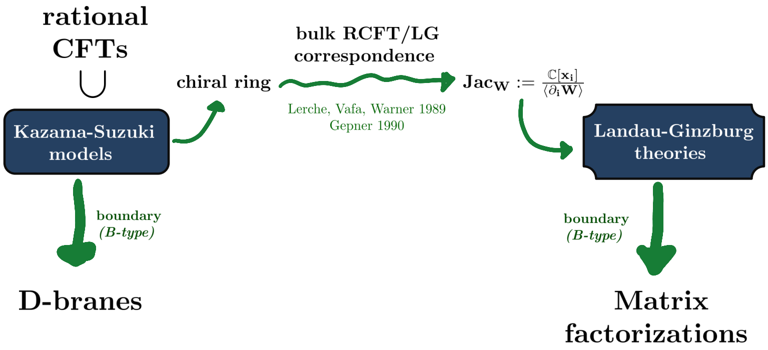

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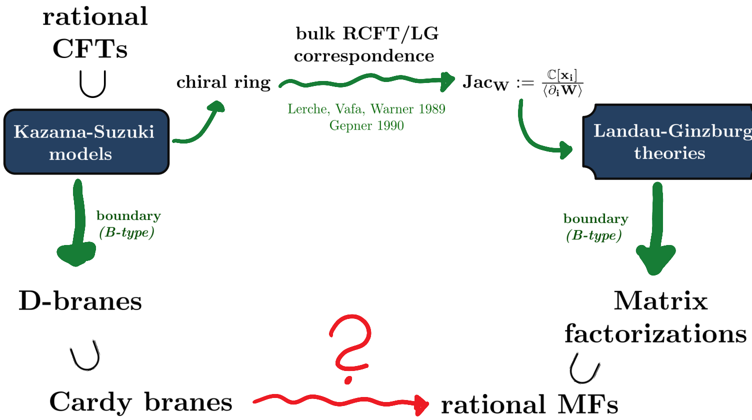
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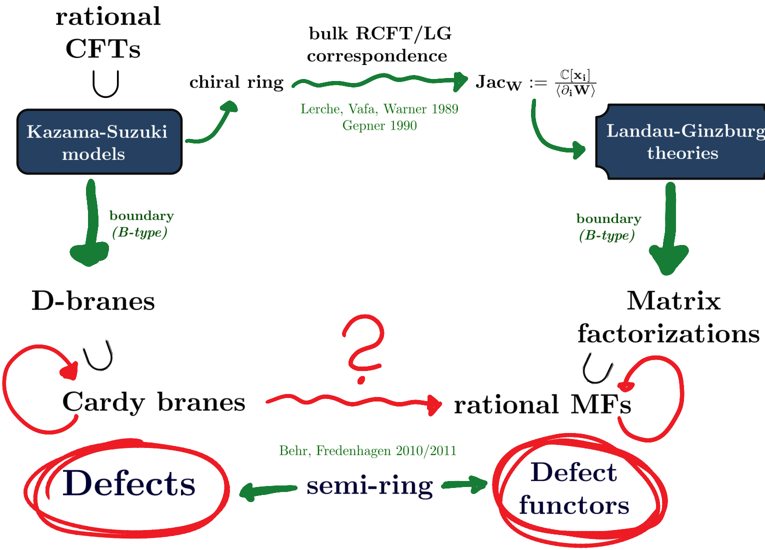

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Behr, Fredenhagen 2010/2011
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A little RCFT background

Bulk correspondence

Introducing (B-type) boundaries

Data for boundary KS models

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WZNW models

Witten, 1984

$\mathbf{G_k}$

- class of CFTs that describe the motion of a string on a group manifold
- G Lie group, $k \in \mathbb{Z}_{>0}$ "level" of the WZNW model
- action is of the form

$$S_{WZNW} = S_{kinetic} + k \cdot S_{WZ}$$

- extraordinary features:
  - ▷ algebra of conserved currents = *affine Lie algebra* $\widetilde{g}_k$
  - ▷ primary fields labeled by *highest weight representations* of $\mathfrak{g}_k$
  - ⇒ finite number of primary fields, i.e. these theories are examples of *rational CFTs*

From WZNW to Kazama-Suzuki models

- Construction: Kazama and Suzuki, 1989
  1. $G_k \xrightarrow{\text{supersymmetrize}} \mathcal{N}=1$ version $\xrightarrow{\text{gauge subgroup}}$ WZNW *coset*
  2. for G/H Hermitean Symmetric Space (HSS) $\Rightarrow$ *KS-model*:

$$\frac{G_k}{H} \times \underbrace{SO(2d)_1}_{\text{Majorana-fermions}}$$

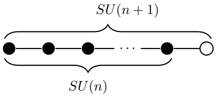
  with:
  - G simple compact Lie group
  - k level of the corresponding affine Lie algebra $\widetilde{g}_k$
  - $H \subset G$ regularly embedded subgroup (i.e. $rk\ G = rk\ H$)
  - $2d = dim\ G - dim\ H$

  Note: the Majorana-fermions are realized in "bosonized form", i.e. as a $so(2d)_1$ WZNW-model
- Motivation: provides a large class of $\mathcal{N}=(2,2)$ rational SCFTs

Grassmannian Kazama-Suzuki models $\mathbf{SU(n+1)_k/U(n)}$

$$\frac{SU(n+k)_1 \times SO(2nk)_1}{SU(n)_{k+1} \times SU(k)_{n+1} \times U(1)} \cong \frac{SU(n+1)_k \times SO(2n)_1}{SU(n)_{k+1} \times U(1)}$$

- Note: we use the diagram embedding



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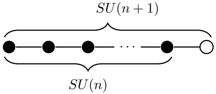

$$i(h,\zeta) = \begin{pmatrix} h\zeta & 0 \\ 0 & \zeta^{-n} \end{pmatrix} \in SU(n+1) \quad h \in SU(n), \zeta \in U(1)$$

Since $i(\xi^{-1}\mathbf{1}, \xi) = \mathbf{1}$ for $\xi^n = 1$, "$H \subset G_k$ " only if we quotient by the $\mathbb{Z}_n$ action:

$$U(n) = \big(SU(n) \times U(1)\big)/\mathbb{Z}_n$$

$\Rightarrow$ field identifications!

$$SU(n+1)_k/U(n) \equiv \frac{SU(n+1)_k \times SO(2n)_1}{SU(n)_{k+1} \times U(1)}$$

- highest weight labels: $(\underbrace{\Lambda}_{su(n+1)_{\boldsymbol{k}}}, \underbrace{\Sigma}_{so(2d)_{\boldsymbol{1}}}; \underbrace{\lambda}_{su(n)_{\boldsymbol{k+1}}}, \underbrace{\mu}_{u(1)_{\boldsymbol{k}^*}})$

  where the $so(2d)_1$ for any d can take values
  - $\Sigma = 0, v$: Neveu-Schwarz sector
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- labels are restricted by Gepner, 1989; Lerche et al., 1989; Moore and Seiberg, 1989
  - identification rules via action of G_{id}, Schellekens and Yankielowicz, 1989, 1990
    generated by the simple current $J_0 = (J_{n+1}, v; J_n, k+n)$

$$(\Lambda, \Sigma; \lambda, \mu) \sim J_0^m(\Lambda, \Sigma; \lambda, \mu) \quad \forall m \in \mathbb{Z}$$

  - selection rules: monodromy charges of the numerator and denominator parts should be equal

$$Q_{J_{n+1}}(\Lambda) + Q_v(\Sigma) \stackrel{!}{=} Q_{J_n}(\lambda) + Q_{k+n}(\mu)$$

    with $Q_J(\phi) = h_J + h_\phi - h_{J\phi} \mod 1$

A little RCFT background

Bulk correspondence

Introducing (B-type) boundaries

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Defect functors

Gepner 1991: KS model $\xrightarrow{\text{choice of } W}$ LG model

Idea: $\{$ring of chiral prim. fields$\}$ $\leftrightarrow$ fusion ring

- chiral primary fields: $h = \frac{q}{2}$ and $\overline{h} = \frac{\overline{q}}{2}$
- OPE of chiral primary fields:

$$\Phi(z)\Upsilon(z') \sim \ldots + \frac{1}{(z-z')^{h_\Phi + h_\Upsilon - h_{\Phi\Upsilon}}}(\Phi\Upsilon)(z) + \ldots$$

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$$\Phi(z)\Upsilon(z) := \lim_{z' \to z} \Phi(z)\Upsilon(z') = \begin{cases} (\Phi\Upsilon)(z), \text{ if } \Phi\Upsilon \text{ is a cpf} \\ 0 \text{ else} \end{cases}$$

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$\Rightarrow$ ring of chiral primary fields

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$$C^{\Lambda_1} \times C^{\Lambda_2} = f^{(su(n+1))\ \Lambda}_{\Lambda_1\ \Lambda_2} f^{(su(n))\ \mathcal{P}\Lambda}_{\mathcal{P}\Lambda_1\ \mathcal{P}\Lambda_2} \delta(Q - Q_1 - Q_2) C^{\Lambda}$$

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- Our paper: explicit computation of the $SU(3)_k/U(2)$ fusion ring via relation generating potential $\Rightarrow W_k(y_1, y_2)$

What is a Landau-Ginzburg theory?

bulk LG-Action: a theory of chiral scalar superfields

$$\mathcal{S}_{LG} = \int d^2z d^4\theta K(\Phi, \overline{\Phi}) + \int d^2z \Big(d^2\theta W(\Phi) + c.c. \Big)$$

with:

- $K(\Phi, \overline{\Phi})$ Kähler potential
- $W(\Phi)$ superpotential
- theory flows to CFT in IR $\Leftrightarrow$ $W(\Phi)$ is quasihomogeneous:

$$W(e^{i\lambda q_i}\Phi_i) = e^{2i\lambda} W(\Phi_i) \quad \forall \lambda \in \mathbb{C}$$

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- *Question*: How do we choose $W(\Phi_i)$?
  Answer: for our purposes (Grassmannian Kazama-Suzuki models), employ Gepner's method, i.e. use the polynomial $W(\Phi_i)$ such that

$$\text{chiral ring of KS model} \mathrel{\widehat{=}} Jac_{W(\Phi_i)} := \frac{\mathbb{C}[\Phi_i]}{\langle \partial_i W \rangle} ,$$

  which implies that a given chiral primary state Λ_{cp} is associated to some explicit polynomial $\widetilde{U}_\Lambda(\Phi_i) \in Jac_{W(\Phi_i)}$.

- A little RCFT background
- Bulk correspondence
- **Introducing (B-type) boundaries**
- Data for boundary KS models
- Boundary LG theory data
- Preliminary version of RCFT/LG boundary correspondnce
- Defect functors

From bulk to boundary KS model

- bulk Hilbert space: "almost diagonal" modular invariant

$$\mathcal{H} = \bigoplus_{[\Lambda,\Sigma;\lambda,\mu]} \mathcal{H}_{[\Lambda,\Sigma;\lambda,\mu]} \otimes \mathcal{H}_{[\Lambda,\Sigma^{+};\lambda,\mu]}$$

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- boundary Hilbert space: via folding trick $\Rightarrow$ theory on upper half plane w/ bdry at the real line $z = \overline{z}$, where we demand B-type gluing conditions:

$$T(z) = \overline{T}(\overline{z}) \quad J(z) = \overline{J}(\overline{z}) \quad G^{\pm}(z) = \eta \overline{G}^{\pm}(\overline{z}) \quad Imz = Im\overline{z}$$

  with: η a sign corresponding to the choice of a spin structure, i.e. of GSO projection

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- B-type D-branes via Cardy construction and factorisation into twisted boundary sectors Fredenhagen, 2003; Ishikawa, 2002; Ishikawa and Tani, 2003, 2004

$$|L,S;l\rangle = \mathcal{N} \sum_{(\Lambda,\Sigma;\lambda,0)\in\mathcal{V}} \frac{\psi^{(n+1)}_{L\Lambda} S^{(so)}_{S\Sigma} \overline{\psi}^{(n)}_{l\lambda}}{\sqrt{S^{(n+1)}_{0\Lambda} S^{(so)}_{0\Sigma} S^{(n)}_{0\lambda}}} |\Lambda,\Sigma;\lambda,0\rangle\rangle$$

Only known solutions: *Cardy branes*

- Severe technical problem: in general, classification and construction of solutions to gluing conditions not known! Notable exception: article by Stanciu [1998]
- Cardy branes are the *maximally symmetric* types of D-brane solutions, i.e. satisfy the much more restrictive gluing conditions

$$W_i(z) = \omega(\overline{W}_i)(\bar{z}),$$

Cardy, 1989

with

- $W_i(z)$ chiral algebra current
- ω outer automorphism of the chiral algebra

$\Rightarrow$ Cardy branes preserve not just the $\mathcal{N} = 2$ symmetry, but the full chiral algebra $\mathcal{A}$ on the boundary!

"LG theory D-branes"

- introducing boundary breaks translation invariance normal to bdry
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- $W \neq 0$: SUSY-variation of $S_{LG}+S_{bdry}$ results in term

$$\delta\left(S_{LG}+S_{bdry}\right)=\frac{i}{2}\int ds\big(\epsilon\overline{\eta}W'-\overline{\epsilon}\eta\overline{W}'\big)\Big|_0^{\pi} \qquad (*)$$

that can not be compensated by contributions to S_{LG} in bulk fields (*Warner problem*) Warner, 1995

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  that can not be compensated by contributions to S_{LG} in bulk fields (*Warner problem*) Warner, 1995
- way out: introduce *boundary fermionic superfield* $\Pi \equiv \Pi(s, \theta^0, \overline{\theta}^0) = \pi(s) + \ldots + \overline{\theta}^0(\mathcal{E}(\Phi) + \ldots)$ with "LG-like" action

$$S_\Pi = -\frac{1}{2}\int dsd^2\theta\overline{\Pi}\Pi\Big|_0^{\pi} - \frac{i}{2}\int dsd\theta\Pi\mathcal{J}(\Phi)_{\overline{\theta}=0}\Big|_0^{\pi} + c.c.$$

  $\Rightarrow$ SUSY-variation of S_Π cancels $(*)$ iff Brunner et al., 2003; Kapustin and Li, 2003; Kontsevich; Orlov, 2003

$$W = \mathcal{J}\cdot\mathcal{E} + const \mathrel{\widehat{=}} \text{matrix factorization!}$$

Main problem: Which MFs are "rational"?

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$\Rightarrow$ Which MFs are "rational", i.e. correspond to Cardy branes in the RCFT?

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Our work: $SU(3)_k/U(2)$

- bulk Hilbert space: $\mathcal{H} = \bigoplus_{[\Lambda,\Sigma;\lambda,\mu]} \mathcal{H}_{[\Lambda,\Sigma;\lambda,\mu]} \otimes \mathcal{H}_{[\Lambda,\Sigma^+;\lambda,\mu]}$
- (B-type) Cardy branes:

$$|\underbrace{L}_{su(3)^{tw}_{k+3}}, \underbrace{S}_{so(2d)_1}; \underbrace{\ell}_{su(2)_{k+4}}\rangle = \mathcal{N} \sum_{(\Lambda,\Sigma;\lambda,0)\in\mathcal{V}} \frac{\psi^{(3)}_{L\Lambda} S^{(so)}_{S\Sigma} \overline{S}^{(2)}_{\ell\lambda}}{\sqrt{S^{(3)}_{0\Lambda} S^{(so)}_{0\Sigma} S^{(2)}_{0\lambda}}} |\Lambda,\Sigma;\lambda,0\rangle\rangle$$

  where $\Psi\ldots$ are the modular S-matrices for the twisted $su(3)_{k+3}$ affine Lie-algebra, while the symbol S stands for the regular modular S-matrices
- $|L,v;\ell\rangle = \overline{|L,0;\ell\rangle} \Rightarrow$ shorthand notation: $|L,\ell\rangle \equiv |L,0;\ell\rangle$
- spectra of (chiral primary) open strings can be computed from

$$\begin{aligned}&\langle L_1, l_1|\widetilde{q}^{\frac{1}{2}(L_0+\overline{L}_0-\frac{c}{12})}|L_2, l_2\rangle_{\text{ch.prim.}}\\ &= \sum_{\Lambda=(\Lambda_1,\Lambda_2)} n_{\Lambda L_2}{}^{L_1} N^{(k+1)\,l_1}_{\Lambda_1 l_2} \chi_{\Lambda,0;\Lambda_1,\Lambda_1+2\Lambda_2}(q)\end{aligned}$$

Ramond-Ramond charges

- B-type D-branes couple only to uncharged RR ground states!
- $SU(3)_k/U(2)$ models:

$$\mathrm{cpf} = \{(\Lambda_1, \Lambda_2), 0; \Lambda_1, \Lambda_1 + 2\Lambda_2)\}$$

$$\xrightarrow{\text{spectral flow}} \mathrm{RGS} \xrightarrow{\text{uncharged}} \mathrm{RGS}_0 = [j] = \{[(j, j), \bar{s}; 2j + 1, 0]\}$$

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- here:

$$\text{ch}_j(|L,l\rangle) = \mathcal{N} \frac{\psi^{(3)}_{L\,(j,j)} S^{so}_{0\bar{s}} S^{(2)}_{l\,2j+1}}{\sqrt{S^{(3)}_{(0,0)(j,j)} S^{so}_{0\bar{s}} S^{(2)}_{0\,2j+1}}}$$

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$$|L,S;\ell\rangle = \mathcal{N} \sum_{(\Lambda,\Sigma;\lambda,0)\in\mathcal{V}} \frac{\psi^{(3)}_{L\Lambda} S^{(so)}_{S\Sigma} \overline{S}^{(2)}_{\ell\lambda}}{\sqrt{S^{(3)}_{0\Lambda} S^{(so)}_{0\Sigma} S^{(2)}_{0\lambda}}} |\Lambda,\Sigma;\lambda,0\rangle\rangle$$

- here:

$$\mathrm{ch}_j(|L,l\rangle) = \mathcal{N} \frac{\psi^{(3)}_{L\,(j,j)} S^{so}_{0\bar{s}} S^{(2)}_{l\,2j+1}}{\sqrt{S^{(3)}_{(0,0)(j,j)} S^{so}_{0\bar{s}} S^{(2)}_{0\,2j+1}}}$$

- basis: in terms of charges of the $|L,0\rangle$ branes

$$\mathrm{ch}_j(|L,l\rangle) = \sum_{L'=0}^{\lfloor\frac{k}{2}\rfloor} \left(N^{(k+1)\,l}_{LL'} - N^{(k+1)\,k+1-l}_{LL'}\right) \mathrm{ch}_j(|L',0\rangle)$$

More structure: Flows and defects

CFT flow rules

Fredenhagen, 2003; Fredenhagen and Schomerus, 2003

flow induced by tachyon $\Psi_* = (\Psi_*^a, \Psi_*^b)$

$$|L,\ell-1\rangle \overset{\Psi_*^a}{\longleftrightarrow} + \; |L,\ell\rangle \overset{\Psi_*^b}{\longleftrightarrow} + \; |L,\ell+1\rangle \rightsquigarrow \begin{cases} \bigoplus_{K=L-1}^{L+1} |K,\ell\rangle & \text{for } L \neq \frac{k}{2} \\ |L-1,\ell\rangle & \text{for } L = \frac{k}{2} \end{cases}$$

Important: Ψ_* has a specific U(1)-R-charge q_{Ψ_*} $(= 1/(k+3))$!

More structure: Flows and defects

Topological defects

here: consider as operators $D_{\Theta} \equiv D_{[(\Lambda_1,\Lambda_2),\Sigma;\lambda,\mu]}$ that

- form a *semi-ring* under "fusion" $*$

$$D_{\Theta_1} * D_{\Theta_2} = \sum_{\Theta} n_{\Theta_1 \Theta 2}{}^{\Theta} D_{\Theta} \qquad (n_{\Theta_1 \Theta 2}{}^{\Theta} \in \mathbb{Z}_{\geq 0})$$

- act on Cardy branes $B_{|L,\ell\rangle}$ (resulting in new Cardy branes)
- **Most important feature**: $\exists$ defect $D_{\Theta_{(1)}}$ that *generates* all Cardy branes from the $|L,0\rangle$ branes via

$$D_{\Theta_{(1)}} * B_{|L,\ell\rangle} = B_{|L,\ell-1\rangle} + B_{|L,\ell+1\rangle}$$

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Basic LG theory data: $\mathbf{hmf^{gr}(W_k)}$

Let R be a *graded* polynomial ring, i.e.

$$R \equiv \mathbb{C}[y_i]^{gr} := \oplus_{i \in \mathbb{Z}_{\geqq 0}} R_i \, ; \, \forall p \in R_i : deg(p) = i$$

with $deg(y_i) = w_i \in \mathbb{N}$. Let $W_k(y_i) \in R_{k+3}$ a quasihomogeneous polynomial. Then

$$W_k(e^{i\lambda q_i} y_i) \stackrel{!}{=} e^{2i\lambda} W_k(y_i) \quad \forall \lambda \in \mathbb{C}^*$$

induces a *U(1)-R-charge grading* $q_{y_i} = 2w_i/(k+3)$.

Basic LG theory data: $\mathbf{hmf^{gr}(W_k)}$

Definition: category $hmf^{gr}(W_k)$

- $Ob(hmf^{gr}(W_k)) := \left\{ {}_R Q \equiv (R, W_k, Q, \sigma, \rho) \right\} / \sim$
  - $Q = \begin{pmatrix} 0 & \mathcal{J} \\ \mathcal{E} & 0 \end{pmatrix}$; $0, \mathcal{J}, \mathcal{E} \in Mat(r \times r; R)$ $(r \in \mathbb{Z}_{>0})$
  - $Q^2 = \begin{pmatrix} \mathcal{J} \cdot \mathcal{E} & 0 \\ 0 & \mathcal{J} \cdot \mathcal{E} \end{pmatrix} = W_k \mathbb{1}_{2r \times 2r}$
  - $\sigma \cdot Q \cdot \sigma = -Q \qquad (\sigma^2 = -\mathbf{1}_{2r \times 2r})$
  - $\rho(\lambda; y_i) Q(e^{i\lambda q_i} y_i) \rho^{-1}(\lambda; y_i) = e^{i\lambda} Q(y_i) \quad \forall \lambda \in \mathbb{C}^*$

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- $Mor(hmf^{gr}(W_k)) := \left\{ H^{i,q}({}_R Q_A, {}_R Q_B) \mid i \in \mathbb{Z}_2, q \in \mathbb{Q} \right\}$
  - $\Phi \in H^{i,q}({}_R Q_A, {}_R Q_b) :\Leftrightarrow$

$$\begin{cases} \sigma_B \Phi \sigma_A = (-1)^{|\Phi|} \Phi \qquad (| \Phi | \in \mathbb{Z}_2) \\ Q_B \Phi - (-1)^{|\Phi|} \Phi Q_A = 0 \quad mod \quad \widetilde{\Phi} = Q_B \widetilde{\Psi} + (-1)^{|\Phi|} \widetilde{\Phi} Q_A \\ \rho_B(\lambda; y_i) \Phi(e^{i\lambda q_i} y_i) \rho_A^{-1}(\lambda; y_i) = e^{i\lambda q} \Phi(y_i) \end{cases}$$

  i.e. this is the definition of some (graded) *cohomology of MFs*
- composition of morphisms: composition in cohomological sense (i.e. naive composition up to exactness)

Equivalence of MFs

Definition

$$Q \sim Q' :\Leftrightarrow \text{ w.l.o.g } rk(Q) \leq rk(Q') \; Q' = \mathcal{U}\left(Q \oplus Q_{triv}^{\oplus m} Q_{triv}^{t^{\oplus \boldsymbol{n}}}\right)\mathcal{U}^{-1}$$

where

- $m, n \in \mathbb{Z}_{\geq 0}$ s.th. $rk(Q) + m + n = rk(Q')$
- $Q_{triv} = \begin{pmatrix} 0 & 1 \\ W_k & 0 \end{pmatrix}$
- $\mathcal{U} \equiv \mathcal{U}(y_i) = \begin{pmatrix} U_1 & 0 \\ 0 & U_2 \end{pmatrix} \in GL(2r' \times 2r'; R) \quad (r' = rk(Q'))$

  i.e. $\mathcal{U}$ is invertible over R:

$$\mathcal{U}\mathcal{U}^{-1} = \mathcal{U}^{-1}\mathcal{U} = \mathbb{1}_{2r' \times 2r'}$$

$\lightning$ severe technical difficulty: equivalences make it hard to guess ''interesting'' MFs!

RR-charges

via *Kapustin-Li formula*: Kapustin and Li, 2004

$$\mathrm{ch}_\phi(Q) = \frac{1}{\sqrt{2}} \mathrm{Res}_{W_k}\Big(\phi \mathrm{Str}\big(\partial_{y_1} Q \partial_{y_2} Q\big)\Big) \ .$$

where $\phi \in Jac_W$ (i.e. some polynomial in y_1 and y_2), Q is a MF and *Str* denotes the supertrace, while the residue is formally defined as

$$\mathrm{Res}_{W_k}(f) = \frac{1}{(2\pi i)^2} \oint \oint \frac{f}{\partial_{y_1} W_k \partial_{y_2} W_k} dy_1 dy_2$$

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- Note: to compare this with the CFT RR charges, we need the explicit "dictionary" between the elements of the CFT and of the LG theory chiral rings:

$$\begin{aligned} \Lambda_{cpf} &\equiv [(\Lambda_1, \Lambda_2), 0; \Lambda_1, \Lambda_1 + 2\Lambda_2] \\ &\widehat{=} \quad \widetilde{U}_{(\Lambda_1,\Lambda_2)}(y_1, y_2) := \sum_{r=0}^{\lfloor \Lambda_1/2 \rfloor} (-1)^r \binom{\Lambda_1 - r}{r} y_1^{\Lambda_1 - 2r} y_2^{\Lambda_2 + r} \end{aligned}$$

More sophisticated structures

Def.: operator $\tau : H^{i,q}({}_R Q_A, {}_R Q_B) : \left(Q_A \xrightarrow{\Phi} Q_B \right) \mapsto \left(Q_{A[-1]} \xrightarrow{\tau\Phi} Q_B \right)$

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Triangulated structure of $hmf^{gr}(W_k)$

- **Def.:** *shift functor* $[1]$
  - $Q = \begin{pmatrix} 0 & \mathcal{J} \\ \mathcal{E} & 0 \end{pmatrix} \mapsto [1]Q \equiv Q[1] := \begin{pmatrix} 0 & -\mathcal{E} \\ -\mathcal{J} & 0 \end{pmatrix}$
  - $\Phi = \underbrace{\begin{pmatrix} \phi_0 & 0 \\ 0 & \phi_1 \end{pmatrix}}_{\in H^{0,q}({}_R Q_A, {}_R Q_B)} \mapsto \Phi[1] := \underbrace{\begin{pmatrix} \phi_1 & 0 \\ 0 & \phi_0 \end{pmatrix}}_{\in H^{0,q}({}_R Q_{A[1]}, {}_R Q_{B[1]})}$

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- **Def.:** *cone functor* c
  - $c\left(Q_A \xrightarrow{\Phi} Q_B\right) \equiv c(\Phi) := \begin{pmatrix} 0 & \mathcal{J}_\Phi \\ \mathcal{E}_\Phi & 0 \end{pmatrix} \equiv \begin{pmatrix} 0 & 0 & \mathcal{J}_B & \tau\phi_0 \\ 0 & 0 & 0 & -\mathcal{E}_A \\ \mathcal{E}_B & \tau\phi_1 & 0 & 0 \\ 0 & -\mathcal{J}_A & 0 & 0 \end{pmatrix}$
  - on diagrams commutative up to exact morphisms A:

$$c\left(\begin{tikzcd} Q_A \arrow[r, "f"] \arrow[d, "g"'] \arrow[dr, squiggly, "a"] & Q_B \arrow[d, "g'"] \\ Q_C \arrow[r, "f'"] & Q_D \end{tikzcd}\right) := \begin{tikzcd} c(f) \arrow[d, "c(g,h;a)"'] \\ c(f') \end{tikzcd} \qquad c(g,h;a)^i := \begin{pmatrix} h^i & a^i \\ 0 & g^{i+1} \end{pmatrix} \ (i \in \mathbb{Z}_2)$$

Uses of triangulated structure

- generate new MFs via $c(Q_A \xrightarrow{\Phi} Q_B)$

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- **Def.:** *distinguished triangles*
  - (TR1) $\forall \Phi \in H^{0,q}({}_R Q_A, {}_R Q_B) \exists$ distinguished $\triangle$

$$Q_A \xrightarrow{\Phi} Q_B \xrightarrow{p(\Phi)} c(\Phi) \xrightarrow{q(\Phi)} Q_{A[1]} \quad p(\Phi) := \begin{pmatrix} \mathbb{1}_B \\ 0 \end{pmatrix} , \; q(\Phi) = \begin{pmatrix} 0 & \mathbb{1}_{A[1]} \end{pmatrix}$$

  - (TR2) if $\mathcal{D}$ as above, then also ALL shifts of $\mathcal{F}$ are distinguished, e.g.

$$\begin{tikzcd}
Q_B \arrow[r, "p(\Phi)"] & c(\Phi) \arrow[r, "q(\Phi)"] \arrow[dr, "p(p(\Phi))"'] & Q_{A[1]} \arrow[r, "\Phi[1]"] & Q_{B[1]} \\
 & & c(p(\Phi)) \arrow[u, dotted, "\exists", "\cong"'] \arrow[ur, "q(p(\Phi))"'] &
\end{tikzcd}$$

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- **Prop.: [Verdier]** *ALL* morphisms $\Phi \in H^{0,q})(Q_A, c(\tau))$ may be obtained as $\Phi = c(g, 0; a)$ for some g, a as in

$$c\left(\begin{tikzcd} Q_A[-1] \arrow[r] \arrow[d, "g"'] \arrow[dr, squiggly, "a"] & 0 \arrow[d] \\ Q_B \arrow[r, "\tau"'] & Q_C \end{tikzcd}\right) \mapsto \begin{tikzcd} Q_A \arrow[d, "c(g,0;a)"'] \\ c(\tau) \end{tikzcd}$$

$\Rightarrow$ may generate complicated cones from simpler MFs!

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Warmup example: $\mathbf{SU(2)_k/U(1)}$ KS model

$$W_k(x) = x^{k+2}$$

- easiest matrix factorizations: polynomial MFs:

$$Q_i = \begin{pmatrix} 0 & x^i \\ x^{k+2-i} & 0 \end{pmatrix} \quad i \in \{1, 2, \ldots, k+2\}$$

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- analysis of spectra and U(1)-R- and RR-charges in both theories

  leads to the association Brunner et al., 2003; Kapustin and Li, 2003

$$|L\rangle \mathrel{\widehat{=}} Q_{L+1}$$

$\Rightarrow$ Complete solution!

Our work: $\mathrm{SU}(3)_k/\mathrm{U}(2)$ models

$$W_k(y_1, y_2) = \prod_{j=0}^{\lfloor\frac{k+1}{2}\rfloor} (y_1^2 - \beta_j y_2) \cdot \begin{cases} y_1 & \text{for } k \text{ even} \\ 1 & \text{for } k \text{ odd} \end{cases}$$

where $\beta_j = 2\big(1 + \cos\left(\pi \frac{2j+1}{d}\right)\big)$

- **via explicit computation of spectra, RR- and $U(1)$-R-charges:**

$$|L, 0\rangle \leftrightarrow Q_{|L,0\rangle} = \begin{pmatrix} 0 & \prod_{j=0}^{L}(y_1^2 - \beta_j y_2) \\ \frac{W_k}{\prod_{j=0}^{L}(y_1^2 - \beta_j y_2)} & 0 \end{pmatrix}$$

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- Only partial match: $|L, 0\rangle$ have no fermions in their self-spectra, unlike all branes $|L, \ell\rangle$ with $\ell > 0$. But all polynomial MFs have no fermions in their self-spectra $\Rightarrow$ need to construct higher-rank MFs!

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  - **specifically for the $SU(3)_k/U(2)$ model: CFT flow rules**

Higher-rank matrix factorization series: $Q_{|L,1\rangle}$

BCFT flow rule

$$|L,0\rangle \overset{\Psi_*}{\longleftrightarrow} + \quad |L,1\rangle \rightsquigarrow \oplus_{K=L-1}^{L+1} |K,0\rangle$$

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"Translating" this into a LG-theory triangle, we obtain:

$$\ldots \to Q_{|L,0\rangle}[1] \xrightarrow{\psi_*} Q_{|L,1\rangle} \to \oplus_{K=L-1}^{L+1} Q_{|K,0\rangle} \to Q_{|L,0\rangle}[2] \to \ldots$$

where we know the MFs colored in green and that the triangle is distinguished for any given morphism ψ_*, whence this allows us to shift the triangle to obtain a candidate for $Q_{|L,1\rangle}$:

$$Q_{|L,1\rangle} \stackrel{?}{=} c\left(\oplus_{K=L-1}^{L+1} Q_{|K,0\rangle}[-1] \to Q_{|L,0\rangle}[1]\right)$$

Higher-rank matrix factorization series: $\mathbf{Q}_{|\mathbf{L},\mathbf{1}\rangle}$

BCFT flow rule

$$\begin{tikzcd} {|L,0\rangle} \arrow[rr, leftrightarrow, bend left, "\Psi_*"] & + & {|L,1\rangle} \rightsquigarrow \oplus_{K=L-1}^{L+1}|K,0\rangle \end{tikzcd}$$

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Explicit analysis shows that there is exactly one possible morphism $\widetilde{\psi}_*$ of the correct U(1)-R-charge, which leads to:

$$|L,1\rangle \mathrel{\widehat{=}} Q_{|L,1\rangle} = c(\oplus_{K=L-1}^{L+1} Q_{|K,0\rangle)[-1]} \xrightarrow{\phi_*} Q_{|L,0\rangle}[1])$$

Brute force Ansatz: SINGULAR!

Via SINGULAR code for the explicit computation of $H^1(Q_A, Q_B)$ for any MFs Q_i (thanks to N. Carqueville for initial code!), we can pursue the brute force Ansatz

$$\begin{array}{c}\text{polynomial MFs}\\ \mathcal{H}^{1,q}(Q_A, Q_B)\\ \downarrow\\ \text{cones of}\\ \text{polynomial MFs}\\ \downarrow\\ \text{cones of cones}\\ \text{of polynomial}\\ \text{MFs}\\ \downarrow\\ \dots\end{array}$$

My code allows to compute the explicit spectra for all such MFs, i.e. we can search for suitable MFs $\Rightarrow$ confirmation of the previously shown MFs, some sporadic mathces for higher label branes $|L, \ell\rangle$ with $\ell > 1$

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A relation between two LG theories. . .

The superpotential of the $SU_k(3)/U(2)$ KS model can be expressed as

$$W_k(y_1, y_2) = \left(x_1^{k+3} + x_2^{k+3}\right)\Big|_{\substack{x_1 + x_2 \mapsto y_1 \\ x_1 x_2 \mapsto y_2}}$$

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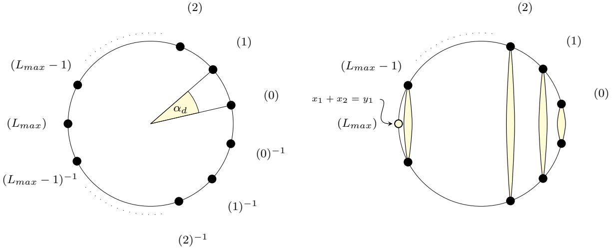
This may be seen from a graphical representation:



$$W(x_i) = \prod_j \left(x_1 - e^{i\alpha_L} x_2\right) \xrightarrow{\text{group symmetric factors}} W(y_i) = y_1 \prod_j \left(y_1^2 - (2 + e^{i\alpha_L} + e^{-i\alpha_L}) y_2\right)$$

A relation between two LG theories...

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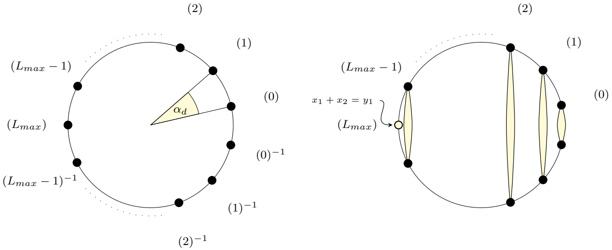

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Also note the $\mathbb{Z}_{k+3}$ *rotation symmetry*!

Pullback and pushforward functors

Definition: Let R and S be two (graded) polynomial rings, $R - mod$ and $S - mod$ the categories of left R- resp. S-modules and homomorphisms, and let $\Phi : R \to S$ be a *ring homomorphism*. Then the *pullback* and *pushforward* functors *along* Φ

$$\Phi^* : R - mod \leftrightarrows S - mod : \Phi_*$$

as follows:

$$\Phi^* : \begin{cases} X \in {}_R M & \mapsto S \otimes_R X \in {}_S M \\ f \in Mor(R - mod) & \mapsto 1_S \underset{R}{\otimes} f \in Mor(S - mod) \end{cases}$$

$$\Phi_* : \quad \text{via} \quad \forall r \in R,\ x \in X, X \in {}_S M\ :\ r.x := \Phi(r).x$$

and analogously for morphisms

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- Note: for suitable choices of Φ, these functors *naturally act on MFs and morphisms of MFs*!

Main result: Defect functor semi-ring

- **Ansatz**: Consider the ring homomprhisms realizing $W_k(y_i) \mapsto \widetilde{W}_k(x_i) = x_1^{k+3} + x_2^{k+3}$ and the morphism that generates the $\mathbb{Z}_{k+3}$ rotation:

$$\iota : R \equiv \mathbb{C}[y_i] \to S \equiv \mathbb{C}[x_i] : \begin{cases} y_1 \mapsto x_1 + x_2 \\ y_2 \mapsto x_1 x_2 \end{cases}$$

$$\gamma_k : S \to S : \begin{cases} x_1 \mapsto x_1 \\ x_2 \mapsto e^{2i\pi/(k+3)} x_2 \end{cases}$$

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- The functor $D_{(1)}$ defined as

Behr and Fredenhagen, 2011

$$\begin{tikzcd} R-mod \arrow[rrr, "D_{(1)}"] \arrow[dr, "\iota^*"'] & & & R-mod \\ & S-mod \arrow[r, "(\gamma_k)^*"] & S-mod \arrow[ur, "\iota_*"'] & \end{tikzcd}$$

Main result: Defect functor semi-ring

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generates a semi-ring of functors $D_{(n)}$, which we name ''defect functors'', according to

$$D_{(1)} \circ D_{(n)} = D_{(n-1)} \oplus D_{(n+1)}$$

New RCFT/LG theory "dictionary" example

With the help of the "defect functors" $D_{(n)}$, we can *generate all "rational" MFs from the simplest MFs* $Q_{|L,0\rangle}$:

$$|L,\ell\rangle \widehat{=} Q_{|L,\ell\rangle} := D_{(\ell)} Q_{|L,0\rangle}$$

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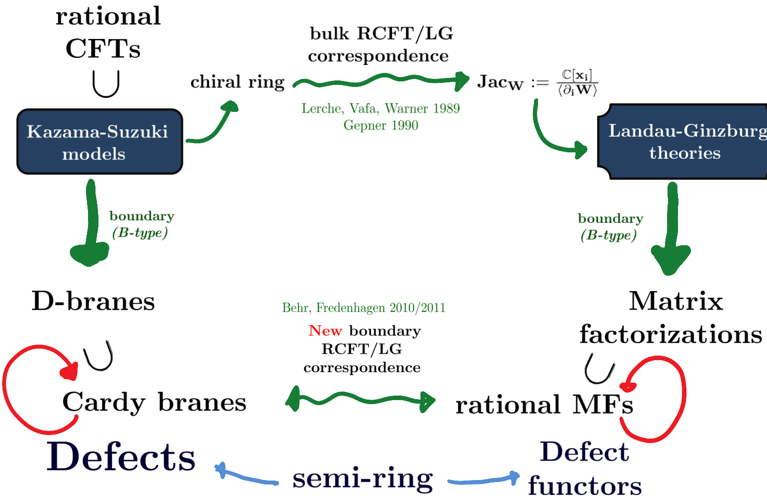
Checks:

- RR charges are automatically correct, since we may act with $D_{(n)}$ on the triangle describing $Q_{|L,1\rangle}$, thereby obtaining

$$c\left(\oplus_{K=L-1}^{L+1} Q_{|K,\ell\rangle}[-1] \xrightarrow{D_{(\ell)}\Phi_{(1)}} Q_{|L,\ell\rangle}[1]\right)$$
$$= D_{(\ell)} D_{(1)} Q_{|L,0\rangle} \cong Q_{|L,\ell-1\rangle} \oplus Q_{|L,\ell+1\rangle}\ ,$$

  which by induction yields the correct result in comparison with the RCFT data
- $\exists$ method (NBSF) to *generate* the correct U(1)-R-charge representations $\rho_{|L,\ell\rangle}$ via $D_{(\ell)}$ directly from the (unambiguously defined) rep $\rho_{|L,0\rangle}$
- explicit computations via SINGULAR for a large number of examples show agreement of spectra including the U(1)-R-charges

Summary



rational CFTs
$\cup$
Kazama-Suzuki models
chiral ring
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Lerche, Vafa, Warner 1989
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$\mathrm{Jac}_W := \frac{\mathbb{C}[x_i]}{\langle \partial_i W \rangle}$
Landau-Ginzburg theories
boundary (*B-type*)
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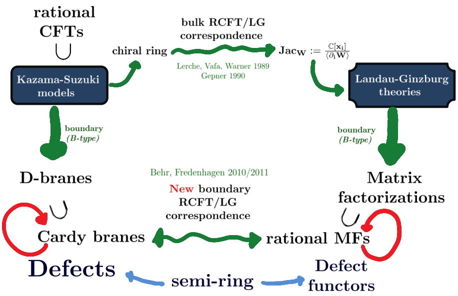
Summary



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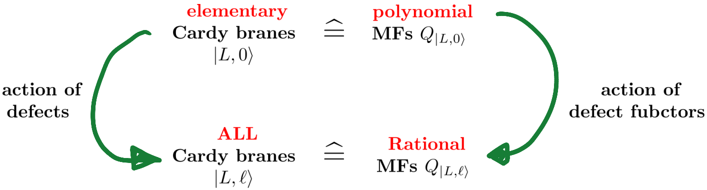


elementary Cardy branes $|L,0\rangle$
$\widehat{=}$
polynomial MFs $Q_{|L,0\rangle}$
action of defects
action of defect fubctors
ALL Cardy branes $|L,\ell\rangle$
$\widehat{=}$
Rational MFs $Q_{|L,\ell\rangle}$

Outlook

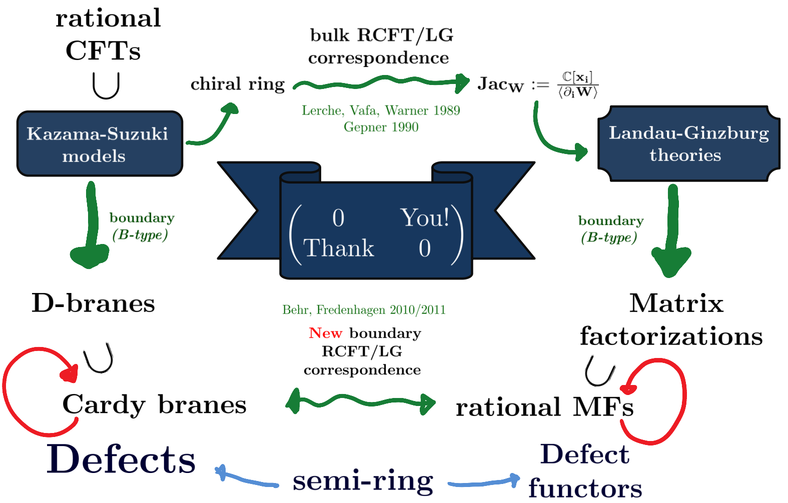
- apply method to other KS models, e.g. the $SU_k(N+1)/U(N)$ Grassmannian models with

$$W_{k,N}(y_1,\ldots,y_{N-1}) := \left(\sum_{i=1}^{N-1} x_i^{k+N+1}\right)\Big|_{s_j(x_i)\mapsto y_j}$$

- $\exists$ deformations of the $SU_k(3)/U(2)$ model that leave the defect functor semi-ring invariant or at least partially preserve it?
- relation to conventional "defect technology" for LG theories: obtain *defefct MFs* via $(\Phi : R \to S)$

$$_R D_S := (\Phi_*, 1)\, {_S}\mathbb{1}_S \qquad {_S}\widetilde{D}_R := (\Phi^*, 1)\, {_R}\mathbb{1}_R$$

  $\Rightarrow$ new insights in the classes of physically relevant topological defects for LG theories!
- potential application: Khovanov-Rozanski link homology computations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