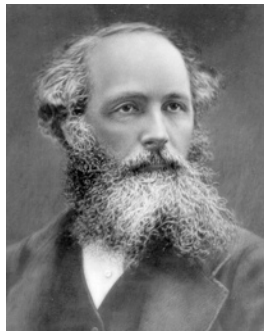


# New $N=5$ gravity duals??

José Miguel Figueroa-O'Farrill



**EMPG Seminar**  
**12 October 2011**



MOTIVATION

A

d

S

/

C

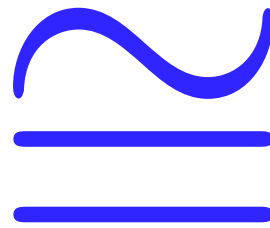
F

T



# Underlying symmetry argument for AdS/CFT:

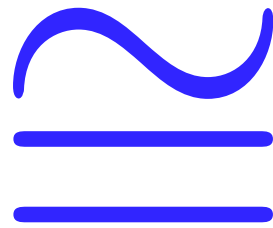
Symmetry superalgebra  
of near horizon limit  
of  $(d-1)$ -brane



Conformal superalgebra  
of  $d$ -dimensional  
CFT

# Underlying symmetry argument for AdS/CFT:

Symmetry superalgebra  
of near horizon limit  
of  $M2$ -brane



Conformal superalgebra  
of  $3$ -dimensional  
CFT

In this talk:  $d=3$

The superalgebra:  $osp(N|4)$

$$\mathfrak{g} = \mathfrak{osp}(N|4)$$

$$\mathfrak{g}_0 = \mathfrak{o}(N) \oplus \mathfrak{sp}(4; \mathbb{R})$$

$$\mathfrak{sp}(4; \mathbb{R}) \cong \mathfrak{o}(3, 2)$$

Lie algebra of  
isometries of  $AdS_4$

$\cong$

Lie algebra of  
conformal isometries  
of  $Mink_3$

This suggests classifying supersymmetric M-theory  
backgrounds with underlying geometry  $AdS_4 \times X^7$

For Freund-Rubin backgrounds,

$$N \geq 4 \quad \implies \quad X^7 = S^7 / \Gamma \quad \exists \Gamma \subset \text{SO}(8)$$

Such backgrounds have been classified:

Morrison+Plesser (1999)

de Medeiros+JMF+Ghadia+Méndez-Escobar (2009)

de Medeiros+JMF (2010)

The group  $\Gamma$  is a **fibred product** of ADE subgroups of  $Sp(1)$ .



TABLE 9. Smooth quotients with  $\mathcal{N} \geq 4$

$A = B = F$	$\tau \in \text{Twist}(A)$	$ A \times_{(A, \tau)} A $	$\mathcal{N}$
$A_1$	1	2	8
$A_{n-1 \geq 2}$	1	$n$	6
$D_{n+2 \geq 4}$	1	$4n$	5
$E_6$	1	24	5
$E_7$	1	48	5
$E_8$	1	120	5
$A_{n-1 \geq 4}$	$1 \neq r \in \mathbb{Z}_n^\times / \langle -1 \rangle$	$n$	4
$D_{n+2 \geq 6}$	$1 \neq r \in \mathbb{Z}_{2n}^\times / \langle -1 \rangle$	$4n$	4
$E_7$	$\mu$	48	4
$E_8$	$\nu$	120	4

TABLE 8. Product subgroups of Spin(4)

$\Gamma$	Order	$\Gamma$	Order
$A_{n-1} \times A_{m-1}$	$nm$	$D_{n+2} \times E_8$	$480n$
$A_{n-1} \times D_{m+2}$	$4nm$	$E_6 \times E_6$	$576$
$A_{n-1} \times E_6$	$24n$	$E_6 \times E_7$	$1152$
$A_{n-1} \times E_7$	$48n$	$E_6 \times E_8$	$2880$
$A_{n-1} \times E_8$	$120n$	$E_7 \times E_7$	$2304$
$D_{n+2} \times D_{m+2}$	$16nm$	$E_7 \times E_8$	$5760$
$D_{n+2} \times E_6$	$96n$	$E_8 \times E_8$	$14400$
$D_{n+2} \times E_7$	$192n$		

TABLE 10. Remaining finite subgroups of Spin(4)

A	B	F	Twist(F)	$ A \times_{(F,\tau)} B $
$A_{kl-1}$	$A_{ml-1}$	$Z_1$	$Z_1^\times / \langle -1 \rangle$	$klm$
$A_{2k-1}$	$D'_{2m+2}$	$Z_2$	$\{1\}$	$8km$
$A_{2k-1}$	$D_{m+2}$	$Z_2$	$\{1\}$	$4km$
$A_{4k-1}$	$D_{2m+3}$	$Z_4$	$\{1\}$	$4k(2m+1)$
$A_{3k-1}$	$E_6$	$Z_3$	$\{1\}$	$24k$
$A_{2k-1}$	$E_7$	$Z_2$	$\{1\}$	$48k$
$D'_{2k+2}$	$D'_{2m+2}$	$Z_2$	$\{1\}$	$32km$
$D_{k+2}$	$D_{m+2}$	$Z_2$	$\{1\}$	$8km$
$D_{2k+2}$	$D_{2m+2}$	$D_4$	$D_6/\sim$	$16km$
$D_{lk+2}$	$D_{lm+2}$	$D_{2l}$	$Z_l^\times / \langle -1 \rangle$	$8klm, (l > 2)$
$D_{(2k+1)+2}$	$D_{(2m+1)+2}$	$Z_4$	$\{1\}$	$4(2k+1)(2m+1)$
$D_{2(2k+1)+2}$	$D_{2(2m+1)+2}$	$2D_4$	$D_6/\sim$	$8(2k+1)(2m+1)$
$D_{l(2k+1)+2}$	$D_{l(2m+1)+2}$	$2D_{2l}$	$Z_{2l}^\times / \langle -1 \rangle$	$4l(2k+1)(2m+1), (l > 2)$
$D_{k+2}$	$D'_{2m+2}$	$Z_2$	$\{1\}$	$16km$
$D'_{2k+2}$	$E_7$	$Z_2$	$\{1\}$	$192k$
$D_{k+2}$	$E_7$	$Z_2$	$\{1\}$	$96k$
$D_{3k+2}$	$E_7$	$D_6$	$\{1\}$	$96k$
$E_6$	$E_6$	$Z_3$	$\{1\}$	$192$
$E_6$	$E_6$	$T$	$\{1\}$	$48$
$E_7$	$E_7$	$Z_2$	$\{1\}$	$1152$
$E_7$	$E_7$	$D_6$	$\{1\}$	$384$
$E_7$	$E_7$	$O$	$\{1\}$	$96$
$E_8$	$E_8$	$I$	$Z_2$	$240$

# Natural Question

Are there M-theory backgrounds with symmetry superalgebra  $osp(N|4)$  which are **not** Freund-Rubin  $AdS_4 \times X$  ?

**NB:** Any such background would be

$\frac{N}{8}$  – supersymmetric.



# Homogeneity Conjecture

“ M-theory backgrounds preserving  $> \frac{1}{2}$  of the supersymmetry are (locally) homogeneous. ”

**Strong form:** the Killing superalgebra already acts locally transitively.

Known to hold for type I and heterotic SUGRAs, and for  $> \frac{3}{4}$ -BPS M-theory and IIB backgrounds.

JMF+Meessen+Philip (2004)

|| JMF+Hackett-Jones+Moutsopoulos (2007)

**Assuming** (the strong form of) the homogeneity conjecture, an M-theory background with symmetry superalgebra  $osp(N|4)$  and  $N > 4$  must be (locally) homogeneous under the action of  $SO(N) \times SO(3,2)$ .

# Programme

(Work in progress with Mara Ungureanu.)

**Classify** (up to local isometry) **11-**dimensional homogeneous lorentzian manifolds with an isometric transitive action of  $SO(N) \times SO(3,2)$  for  $N > 4$ .

**Identify** those which give rise to eleven-dimensional supergravity backgrounds.

**Determine** how much supersymmetry is preserved by the backgrounds.

HOMOGENEOUS  
LORENTZIAN  
MANIFOLDS



A (locally) homogeneous lorentzian manifold is (locally) isometric to  $G/H$ . We are interested in  $G=SO(N)\times SO(3,2)$  for  $N>4$  and  $H$  a closed subgroup of  $G$ . Since  $\dim(G/H)=11$ ,

$$\dim H + 1 = \dim SO(N).$$

These are characterised by the following algebraic data:

$$\mathfrak{h} \subset \mathfrak{g} = \mathfrak{so}(N) \oplus \mathfrak{so}(3,2) \quad \dim \mathfrak{h} = \binom{N}{2} - 1$$

fixing a lorentzian inner product on  $\mathfrak{g}/\mathfrak{h}$ .

$$\mathfrak{h} \subset \mathfrak{g} = \mathfrak{so}(N) \oplus \mathfrak{so}(3, 2)$$

(Lie algebraic) **Goursat Lemma:**

Every such  $\mathfrak{h}$  is a fibred product of two subalgebras  $\mathfrak{h}_1 \subset \mathfrak{so}(N)$  and  $\mathfrak{h}_2 \subset \mathfrak{so}(3, 2)$  over a common quotient.

$$\begin{array}{ccc} \mathfrak{h} & \longrightarrow & \mathfrak{h}_1 \\ \downarrow & & \downarrow \\ \mathfrak{h}_2 & \longrightarrow & \mathfrak{q} \end{array}$$

# Let's meet the Lie subalgebras of $\mathfrak{so}(3,2)$ !

Patera+Sharp+Winternitz+Zassenhaus (1977)

TABLE IV. Subalgebras of the similitude algebra.

Name and range of parameters	Generators	Isomorphism class	Normalizer	Invariants	Maximal subalgebras
$a_{7,1}$	$F; K_1, K_2, L_3, P_0, P_1, P_2$	$LSim(2, 1)$	self	$(K_1P_2 - K_2P_1 - L_3P_0)^2 / (P_0^2 - P_1^2 - P_2^2)$	$a_{6,1}, a_{6,2}, a_{5,7}, a_{4,4}$
$a_{6,1}$	$; K_1, K_2, L_3, P_0, P_1, P_2$	$LE(2, 1)$	$a_{7,1}$	$P_0^2 - P_1^2 - P_2^2, K_1P_2 - K_2P_1 - L_3P_0$	$a_{5,1}, a_{4,3}, a_{3,24}$
$a_{6,2}$	$F, K_2; K_1 - L_3, P_0, P_1, P_2$	$F \square A_{5,30}^0$	self	none	$a_{5,1}, a_{5,2}, a_{5,3}, a_{5,4}^a, a_{5,5}, a_{5,6}, a_{5,8}$
$a_{5,1}$	$K_2; L_3 - K_1, P_0, P_1, P_2$	$A_{5,30}^0$	$a_{6,2}$	$P_0^2 - P_1^2 - P_2^2$	$a_{4,1}, a_{4,7}, a_{4,14}$
$a_{5,2}$	$F - K_2; -K_1 + L_3, P_0, P_1, P_2$	$A_{5,30}^{-1}$	$a_{6,2}$	$P_0 - P_2$	$a_{4,2}, a_{4,7}, a_{4,13}$
$a_{5,3}$	$F + K_2; -K_1 + L_3, P_0, P_1, P_2$	$A_{5,30}^1$	$a_{6,2}$	$(P_0^2 - P_1^2 - P_2^2) / (P_0 - P_2)$	$a_{4,2}, a_{4,7}, a_{4,12}^{\xi}, \bar{a}_{4,18}$
$a_{5,4}^a$ $a \neq 0, \pm 1$	$F + aK_2; -K_1 + L_3, P_0, P_1, P_2$	$A_{5,30}^{1/a}$	$a_{6,2}$	$(P_0 - P_2)^{2/(1+a)} / (P_0^2 - P_1^2 - P_2^2)$	$a_{4,7}, a_{4,10}^{\alpha}, a_{4,15}^a, a_{4,16}^{\xi} (a = \frac{1}{2})$
$a_{5,5}$	$F, L_3 - K_1; P_0, P_1, P_2$	$A_{5,32}^0$	$a_{6,2}$	$(P_0^2 - P_1^2 - P_2^2) / (P_0 - P_2)^2$	$a_{4,7}, a_{4,8}^{\xi}, a_{4,9}, \bar{a}_{4,14}$
$a_{5,6}$	$F, K_2; P_0, P_1, P_2$	$A_{5,33}^{1/2, 1/2}$	self	$(P_0^2 - P_1^2 - P_2^2) / P_1^2$	$a_{4,1}, a_{4,2}, a_{4,5}, a_{4,6}, a_{4,9}, a_{4,10}^b$
$a_{5,7}$	$F, L_3; P_0, P_1, P_2$	$A_{3,35}^{0,1}$	self	$(P_0^2 - P_1^2 - P_2^2) / P_0^2$	$a_{4,3}, a_{4,9}, a_{4,11}, a_{4,17}, a_{3,5}$
$a_{5,8}$	$F, K_2; L_3 - K_1, P_0 - P_2, P_1$	$A_{5,36}$	self	$[(L_3 - K_1)P_1 / (P_0 - P_2)] + K_2 - F$	$a_{4,5}, \bar{a}_{4,5}, a_{4,13}, a_{4,14}, \bar{a}_{4,14}, a_{4,15}^b, \bar{a}_{4,18}$

TABLE IV. (continued)

Name and range of parameters	Generators	Isomorphism class	Normalizer	Invariants	Maximal subalgebras
$a_{4,1}$	$P_1 \oplus \{K_2; P_0, P_2\}$	$A_1 \oplus A_{3,4}$	$a_{5,6}$	$P_1, P_0^2 - P_2^2$	$a_{3,1}, a_{3,2}, a_{3,13}, a_{3,14}^{\epsilon}$
$a_{4,2}$	$P_0 - P_2 \oplus \{F - K_2; P_0 + P_2, P_1\}$	$A_1 \oplus A_{3,5}^{1/2}$	$a_{5,6}$	$P_0 - P_2, (P_0 + P_2) / P_1^2$	$a_{3,1}, a_{3,6}, a_{3,7}, a_{3,16}, a_{3,17}^{\epsilon}$
$a_{4,3}$	$P_0 \oplus \{L_3; P_1, P_2\}$	$A_1 \oplus A_{3,6}$	$a_{5,7}$	$P_0, P_1^2 + P_2^2$	$a_{3,1}, a_{3,21}, a_{3,22}^{\epsilon}, a_{2,3}$
$a_{4,4}$	$F \oplus \{K_1, K_2, L_3\}$	$A_1 \oplus A_{3,8}$	self	$F, K_1^2 + K_2^2 - L_3^2$	$a_{3,3}, a_{3,24}, a_{2,7}$
$a_{4,5}$	$\{K_2; P_0 - P_2\} \oplus \{F - K_2; P_1\}$	$A_2 \oplus A_2$	self	none	$a_{3,2}, \tilde{a}_{3,3}, a_{3,4}, a_{3,7}, \tilde{a}_{3,10}, a_{3,15}, a_{3,16}, a_{3,19}^{\epsilon}$
$\tilde{a}_{4,5}$	$\{F; P_0 - P_2\} \oplus \{F - K_2, L_3 - K_1\}$	$A_2 \oplus A_2$	self	none	$\tilde{a}_{3,2}, a_{3,3}, a_{3,4}, \tilde{a}_{3,7}, a_{3,10}, \tilde{a}_{3,15}, \tilde{a}_{3,16}, \tilde{a}_{3,19}$
$a_{4,6}$	$\{F; K_2; P_0 - P_2\} \oplus \{F - K_2; P_0 + P_2\}$	$A_2 \oplus A_2$	self	none	$a_{3,4}, a_{3,6}, a_{3,12}, a_{3,13}, a_{3,20}^{\epsilon}$
$a_{4,7}$	$L_3 - K_1, P_0 + P_2; P_0 - P_2, P_1$	$A_{4,1}$	$a_{6,2}$	$P_0 - P_2, P_0^2 - P_1^2 - P_2^2$	$a_{3,1}, a_{3,8}^{\epsilon}, \bar{a}_{3,25}$
$a_{4,8}^{\epsilon}$ $\epsilon = 1 [\epsilon = \pm 1]$	$F + \epsilon(L_3 - K_1); P_0, P_1, P_2$	$A_{4,4}$	$a_{5,5}$	$(P_0 - P_2)^2 / (P_0^2 - P_1^2 - P_2^2), (P_0 - P_2)^{\epsilon} \exp[P_1 / (P_2 - P_0)]$	$a_{3,1}, \tilde{a}_{3,9}^{\epsilon}$
$a_{4,9}$	$F; P_0, P_1, P_2$	$A_{3,5}^{1/2}$	$a_{7,1}$	$P_1 / P_0, P_2 / P_0$	$a_{3,1}, \tilde{a}_{3,10}, a_{3,11}, a_{3,12}$
$a_{4,10}^b$ $b > 0, \neq 1$	$F - bK_2; P_0, P_1, P_2$	$A_{4,5}^{1/(1-b), (1+b)/(1-b)}$	$a_{5,6}$	$P_1^2 / (P_0^2 - P_1^2 - P_2^2), P_1^{2b} (P_0 - P_2) / (P_0 + P_2)$	$a_{3,1}, a_{3,15} (b = 2), a_{3,19}, a_{3,20}$
$a_{4,11}^b$ $b > 0 [b \neq 0]$	$F + bL_3; P_0, P_1, P_2$	$A_{3,6}^{1/b, 1/b}$	$a_{5,7}$	$P_0^2 / (P_1^2 + P_2^2), (P_1^2 + P_2^2)^b / (P_1 + iP_2)^i (P_1 - iP_2)^{-i}$	$a_{3,1}, a_{3,23}^b, a_{3,17}^b$
$a_{4,12}^{\epsilon}$ $\epsilon = 1^* [\epsilon = \pm 1]$	$F + K_2 + \epsilon(P_0 + P_2); -K_1 + L_3, P_0 - P_2, P_1$	$A_{4,7}$	$a_{5,3}$	none	$a_{3,17}, \bar{a}_{3,25}$
$a_{4,13}$	$F - K_2; P_0 - P_2, -K_1 + L_3, P_1$	$A_{4,8}$	$a_{5,8}$	$P_0 - P_2, (P_0 - P_2)(F - K_2) - P_1(-K_1 + L_3)$	$a_{3,7}, \tilde{a}_{3,7}, \bar{a}_{3,25}$
$a_{4,14}$	$K_2, P_1; -K_1 + L_3, P_0 - P_2$	$A_{4,9}^0$	$a_{5,8}$	none	$a_{3,2}, a_{3,9}^{\epsilon}, a_{3,10}, \bar{a}_{3,25}$
$\tilde{a}_{4,14}$	$F, -K_1 + L_3; P_1, P_0 - P_2$	$A_{4,9}^0$	$a_{5,8}$	none	$\tilde{a}_{3,2}, \tilde{a}_{3,9}^{\epsilon}, \tilde{a}_{3,10}, \bar{a}_{3,25}$
$a_{4,15}^b$ $[b \neq 0, \pm 1]$	$F + bK_2; -K_1 + L_3, P_0 - P_2, P_1$	$A_{4,9}^h$ $h = b,  b  < 1$ $h = b^{-1},  b  > 1$	$a_{5,8}$	none	$a_{3,15} (b = -2), \tilde{a}_{3,15} (b = -\frac{1}{2}), a_{3,19}, \tilde{a}_{3,19}, \bar{a}_{3,25}$

TABLE IV. (continued)

Name and range of parameters	Generators	Isomorphism class	Normalizer	Invariants	Maximal subalgebras
$a_{4,16}^\epsilon$ $\epsilon = 1 * [\epsilon = \pm 1]$	$F + \frac{1}{2}K_2; -K_1 + L_3$ $+ \epsilon(P_0 + P_2), P_0 - P_2, P_1$	$A_{4,9}^{1/2}$	$a_{5,4}^{1/2}$	none	$a_{3,8}^\epsilon, a_{3,18}^\epsilon, a_{3,19}^{1/2}$
$a_{4,17}$	$F, L_3; P_1, P_2$	$A_{4,12}$	self	none	$a_{3,11}, a_{3,21},$ $a_{3,23}^\epsilon, a_{2,7}$
$\bar{a}_{4,18} = b_{4,5}$	$F + K_2; -K_1 + L_3,$ $P_0 - P_2, P_1$	$A_{4,9}^1$	$b_{7,1}$	none	$a_{3,16}, \bar{a}_{3,16},$ $\bar{a}_{3,25}$
$a_{3,1}$	$P_0, P_1, P_2$	$3A_1$	$a_{7,1}$	$P_0, P_1, P_2$	$a_{2,2}, a_{2,4}, a_{2,5}$
$a_{3,2}$	$P_1 \oplus \{K_2; P_0 - P_2\}$	$A_1 \oplus A_2$	$a_{4,5}$	$P_1$	$\bar{a}_{2,1}, a_{2,2}, a_{2,11},$ $a_{2,12}^\epsilon$
$\bar{a}_{3,2}$	$-K_1 + L_3 \oplus$ $\{F; P_0 - P_2\}$	$A_1 \oplus A_2$	$\bar{a}_{4,5}$	$-K_1 + L_3$	$a_{2,1}, \bar{a}_{2,2}, \bar{a}_{2,11},$ $\bar{a}_{2,12}^\epsilon$
$a_{3,3}$	$F \oplus \{K_2; -K_1 + L_3\}$	$A_1 \oplus A_2$	self	$F$	$a_{2,1}, a_{2,6}, a_{2,10},$ $a_{2,14}^\epsilon, \bar{a}_{2,15}^\epsilon$
$\bar{a}_{3,3}$	$K_2 \oplus \{F; P_1\}$	$A_1 \oplus A_2$	self	$K_2$	$\bar{a}_{2,1}, a_{2,6}, \bar{a}_{2,10},$ $\bar{a}_{2,14}^\epsilon, a_{2,15}$
$a_{3,4}$	$F - K_2 \oplus$ $\{F + K_2; P_0 - P_2\}$	$A_1 \oplus A_2$	self	$F - K_2$	$a_{2,6}, a_{2,8}, a_{2,11},$ $\bar{a}_{2,11}, a_{2,18}^\epsilon, \bar{a}_{2,21}$
$a_{3,5}$	$L_3 \oplus \{F; P_0\}$	$A_1 \oplus A_2$	self	$L_3$	$a_{2,3}, a_{2,7}, a_{2,13},$ $a_{2,17}^\epsilon$
$a_{3,6}$	$P_0 - P_2 \oplus$ $\{F - K_2; P_0 + P_2\}$	$A_1 \oplus A_2$	$a_{4,6}$	$P_0 - P_2$	$a_{2,5}, a_{2,8}, a_{2,19},$ $\bar{a}_{2,21}$
$a_{3,7}$	$P_0 - P_2 \oplus$ $\{F - K_2; P_1\}$	$A_1 \oplus A_2$	$a_{4,5}$	$P_0 - P_2$	$a_{2,2}, a_{2,8}, a_{2,15},$ $a_{2,16}^\epsilon$
$\bar{a}_{3,7}$	$P_0 - P_2 \oplus$ $\{F - K_2; -K_1 + L_3\}$	$A_1 \oplus A_2$	$\bar{a}_{4,5}$	$P_0 - P_2$	$\bar{a}_{2,2}, a_{2,8}, \bar{a}_{2,15},$ $\bar{a}_{2,16}^\epsilon$
$a_{3,8}^\epsilon$ $\epsilon = 1 * [\epsilon = \pm 1]$	$-K_1 + L_3 + \epsilon(P_0 + P_2),$ $P_1; P_0 - P_2$	$A_{3,1}$	$a_{5,4}^{1/2}$	$P_0 - P_2$	$a_{2,2}, a_{2,9}^\epsilon$
$a_{3,9}^\epsilon$ $\epsilon = 1 [\epsilon = \pm 1]$	$K_2 - \epsilon P_1;$ $-K_1 + L_3, P_0 - P_2$	$A_{3,2}$	$a_{4,14}$	$(P_0 - P_2)^{-\epsilon} \exp\{(-K_1 + L_3)/(P_0 - P_2)\}$	$\bar{a}_{2,2}, a_{2,12}^\epsilon$
$\bar{a}_{3,9}^\epsilon$ $\epsilon = 1 [\epsilon = \pm 1]$	$F + \epsilon(-K_1 + L_3);$ $P_1, P_0 - P_2$	$A_{3,2}$	$\bar{a}_{4,14}$	$(P_0 - P_2)^{-\epsilon} \exp\{P_1/(P_0 - P_2)\}$	$a_{2,2}, \bar{a}_{2,12}^\epsilon$
$a_{3,10}$	$K_2; -K_1 + L_3,$ $P_0 - P_2$	$A_{3,3}$	$a_{5,8}$	$(-K_1 + L_3)/(P_0 - P_2)$	$\bar{a}_{2,2}, a_{2,10}, a_{2,11}$
$\bar{a}_{3,10}$	$F; P_1, P_0 - P_2$	$A_{3,3}$	$a_{5,8}$	$P_1/(P_0 - P_2)$	$a_{2,2}, \bar{a}_{2,10}, \bar{a}_{2,11}$
$a_{3,11}$	$F; P_1, P_2$	$A_{3,3}$	$a_{4,17}$	$P_1/P_2$	$a_{2,4}, \bar{a}_{2,10}$
$a_{3,12}$	$F; P_0, P_2$	$A_{3,3}$	$a_{4,6}$	$P_0/P_2$	$a_{2,5}, \bar{a}_{2,10}, \bar{a}_{2,11},$ $a_{2,13}$
$a_{3,13}$	$K_2; P_0, P_2$	$A_{3,4}$	$a_{5,6}$	$P_0^2 - P_2^2$	$a_{2,5}, a_{2,11}$
$a_{3,14}^\epsilon$ $\epsilon = 1 [\epsilon = \pm 1]$	$K_2 - \epsilon P_1; P_0, P_2$	$A_{3,4}$	$a_{4,1}$	$P_0^2 - P_2^2$	$a_{2,5}, a_{2,12}^\epsilon$
$a_{3,15}$	$F - 2K_2; P_1,$ $P_0 - P_2$	$A_{3,4}$	$a_{4,5}$	$P_1(P_0 - P_2)$	$a_{2,2}, \bar{a}_{2,14}, a_{2,18}^2$

TABLE IV. (continued)

Name and range of parameters	Generators	Isomorphism class	Normalizer	Invariants	Maximal subalgebras
$\tilde{a}_{3,15}$	$F - \frac{1}{2}K_2;$ $-K_1 + L_3, P_0 - P_2$	$A_{3,4}$	$\tilde{a}_{4,5}$	$(-K_1 + L_3)(P_0 - P_2)$	$\tilde{a}_{2,2}, a_{2,14}^{1/2}, a_{2,16}^{1/2}$
$a_{3,16}$	$F + K_2; P_1,$ $P_0 - P_2$	$A_{3,5}^{1/2}$	$a_{5,6}$	$P_1^2 / (P_0 - P_2)$	$a_{2,2}, a_{2,15}, \bar{a}_{2,21}$
$\tilde{a}_{3,16}$	$F + K_2; -K_1 + L_3, P_0 - P_2$	$A_{3,5}^{1/2}$	$\sim a_{5,6}$	$(-K_1 + L_3)^2 / (P_0 - P_2)$	$\tilde{a}_{2,2}, \tilde{a}_{2,15}^1, \bar{a}_{2,21}$
$a_{3,17}^\epsilon$ $\epsilon = 1 * [\epsilon = \pm 1]$	$F + K_2 + \epsilon(P_0 + P_2);$ $P_0 - P_2, P_1$	$A_{3,5}^{1/2}$	$a_{4,2}$	$P_1^2 / (P_0 - P_2)$	$a_{2,2}, a_{2,16}^\epsilon, a_{2,19}^\epsilon$
$a_{3,18}^\epsilon$ $\epsilon = 1 [\epsilon = \pm 1]$	$F + \frac{1}{2}K_2; -K_1 + L_3$ $+ \epsilon(P_0 + P_2), P_0 - P_2$	$A_{3,5}^{1/2}$	self	$(P_0 - P_2) / (-K_1 + L_3 + \epsilon(P_0 + P_2))^3$	$a_{2,9}^\epsilon, a_{2,18}^{1/2}, a_{2,20}^\epsilon$
$a_{3,19}^\epsilon$ $c \neq 0, \pm 1, -2$	$F + cK_2; P_1,$ $P_0 - P_2$	$A_{3,5}^h, h = 1 + c, -2 < c < 0$ $h = (1 + c)^{-1}, c < -2 \text{ or } c > 0$	$a_{4,5}$	$(P_0 - P_2)P_1^{1-c}$	$a_{2,2}, \tilde{a}_{2,14}^c, a_{2,18}^\epsilon$
$\tilde{a}_{3,19}^\epsilon$ $c \neq 0, -\frac{1}{2}, \pm 1$	$F + cK_2;$ $-K_1 + L_3, P_0 - P_2$	$A_{3,5}^h, h = \frac{c}{c+1}, c > -\frac{1}{2}$ $h = \frac{c+1}{c}, c < -\frac{1}{2}$	$\tilde{a}_{4,5}$	$(P_0 - P_2) / (L_3 - K_1)^{-1+c} / c$	$\tilde{a}_{2,2}, a_{2,14}^c, a_{2,18}^\epsilon$
$a_{3,20}^\epsilon$ $c > 0, c \neq 1$	$F + cK_2; P_0, P_2$	$A_{3,5}^{(1-c)/(1+c)}$	$a_{4,6}$	$(P_0 + P_2) / (P_0 - P_2)^{-1+c} / (1+c)$	$a_{2,5}, a_{2,18}^\epsilon$
$a_{3,21}$	$L_3; P_1, P_2$	$A_{3,6}$	$a_{5,7}$	$P_1^2 + P_2^2$	$a_{2,4}, \bar{a}_{1,10}$
$a_{3,22}^\epsilon$ $\epsilon = 1 [\epsilon = \pm 1]$	$L_3 + \epsilon P_0; P_1, P_2$	$A_{3,6}$	$a_{4,3}$	$P_1^2 + P_2^2$	$a_{2,4}, a_{1,6}^\epsilon$
$a_{3,23}^\epsilon$ $c > 0 [c \neq 0]$	$F + cL_3; P_1, P_2$	$A_{3,7}^{1/c}$	$a_{4,17}$	$(P_1^2 + P_2^2)^c / (P_1 + iP_2)^i (P_1 - iP_2)^{-i}$	$a_{2,4}, a_{1,7}^\epsilon$
$a_{3,24}$	$;K_1, K_2, L_3$	$A_{3,8}$	$a_{4,4}$	$K_1^2 + K_2^2 - L_3^2$	$a_{2,10}, \bar{a}_{1,10}$
$\bar{a}_{3,25} = \bar{b}_{3,2}$	$-K_1 + L_3; P_1;$ $P_0 - P_2$	$A_{3,1}$	$b_{7,1}$	$P_0 - P_2$	$a_{2,2}, \bar{a}_{2,2}$
$a_{2,1}$	$F, -K_1 + L_3$	$2A_1$	$a_{3,3}$	$F, -K_1 + L_3$	$a_{1,1}, \tilde{a}_{1,2}, a_{1,4}^\epsilon$
$\tilde{a}_{2,1}$	$K_2, P_1$	$2A_1$	$\tilde{a}_{3,3}$	$K_2, P_1$	$\tilde{a}_{1,1}, a_{1,2}, \tilde{a}_{1,4}^\epsilon$
$a_{2,2}$	$P_1, P_0 - P_2$	$2A_1$	$a_{6,2}$	$P_1, P_0 - P_2$	$a_{1,2}, \bar{a}_{1,11}$
$\tilde{a}_{2,2}$	$-K_1 + L_3;$ $P_0 - P_2$	$2A_1$	$b_{6,2}$	$-K_1 + L_3,$ $P_0 - P_2$	$\tilde{a}_{1,2}, \bar{a}_{1,11}$
$a_{2,3}$	$L_3, P_0$	$2A_1$	$a_{3,5}$	$L_3, P_0$	$a_{1,3}, a_{1,6}^\epsilon, \bar{a}_{1,10}$
$a_{2,4}$	$P_1, P_2$	$2A_1$	$a_{5,7}$	$P_1, P_2$	$a_{1,2}$
$a_{2,5}$	$P_0, P_2$	$2A_1$	$a_{5,6}$	$P_0, P_2$	$a_{1,2}, a_{1,3}, \bar{a}_{1,11}$
$a_{2,6}$	$F, K_2$	$2A_1$	self	$F, K_2$	$a_{1,1}, \tilde{a}_{1,1}, a_{1,8}^\epsilon,$ $\bar{a}_{1,12}$
$a_{2,7}$	$F, L_3$	$2A_1$	self	$F, L_3$	$a_{1,1}, a_{1,7}^\epsilon, \bar{a}_{1,10}$
$a_{2,8}$	$F - K_2,$ $P_0 - P_2$	$2A_1$	$a_{3,4}$	$F - K_2, P_0 - P_2$	$a_{1,3}, \bar{a}_{1,11}, \bar{a}_{1,12}$
$a_{2,9}^\epsilon$ $\epsilon = 1 [\epsilon = \pm 1]$	$L_3 - K_1 + \epsilon(P_0 + P_2),$ $P_0 - P_2$	$2A_1$	$a_{3,18}^\epsilon$	$L_3 - K_1 + \epsilon(P_0 + P_2),$ $P_0 - P_2$	$a_{1,5}, \bar{a}_{1,11}$

TABLE IV. (continued)

Name and range of parameters	Generators	Isomorphism class	Normalizer	Invariants	Maximal subalgebras
$a_{2,10}$	$K_2; -K_1 + L_3$	$A_2$	$a_{3,3}$	none	$\bar{a}_{1,1}, \bar{a}_{1,2}$
$\tilde{a}_{2,10}$	$F; P_1$	$A_2$	$\tilde{a}_{3,3}$	none	$a_{1,1}, a_{1,2}$
$a_{2,11}$	$K_2; P_0 - P_2$	$A_2$	$a_{4,5}$	none	$\bar{a}_{1,1}, \bar{a}_{1,11}$
$\tilde{a}_{2,11}$	$F; P_0 - P_2$	$A_2$	$\tilde{a}_{4,5}$	none	$a_{1,1}, \bar{a}_{1,11}$
$a_{2,12}^\epsilon$ $\epsilon = \pm 1$ [ $\epsilon = \pm 1$ ]	$K_2 - \epsilon P_1; P_0 - P_2$	$A_2$	$a_{3,2}$	none	$\bar{a}_{1,4}, \bar{a}_{1,11}$
$\tilde{a}_{2,12}^\epsilon$ $\epsilon = \pm 1$ [ $\epsilon = \pm 1$ ]	$F + \epsilon(-K_1 + L_3); P_0 - P_2$	$A_2$	$\tilde{a}_{3,2}$	none	$a_{1,4}, \bar{a}_{1,11}$
$a_{2,13}$	$F; P_0$	$A_2$	$a_{3,5}$	none	$a_{1,1}, a_{1,3}$
$a_{2,14}^d$ $d > 0, d \neq 1$ [ $d \neq 0, \pm 1$ ]	$F - dK_2; -K_1 + L_3$	$A_2$	$a_{3,3}$	none	$\bar{a}_{1,2}, a_{1,8}^d$
$\tilde{a}_{2,14}^d$ $d > 0, d \neq 1$ [ $d \neq 0, \pm 1$ ]	$F - dK_2; P_1$	$A_2$	$\tilde{a}_{3,3}$	none	$a_{1,2}, a_{1,8}^d$
$a_{2,15}$	$F - K_2; P_1$	$A_2$	$a_{4,5}$	none	$a_{1,2}, a_{1,12}$
$\tilde{a}_{2,15}^\epsilon$ $\epsilon = \pm 1$ [ $\epsilon = \pm 1$ ]	$F - \epsilon K_2; -K_1 + L_3$	$A_2$	$\tilde{a}_{4,5}$	none	$\bar{a}_{1,2}, \bar{a}_{1,12}$
$a_{2,16}^\epsilon$ $\epsilon = \pm 1$ * [ $\epsilon = \pm 1$ ]	$F - K_2 + \epsilon(P_0 - P_2); P_1$	$A_2$	$a_{3,7}$	none	$a_{1,2}, a_{1,9}^\epsilon$
$\tilde{a}_{2,16}^\epsilon$ $\epsilon = \pm 1$ * [ $\epsilon = \pm 1$ ]	$F - K_2 + \epsilon(P_0 - P_2); -K_1 + L_3$	$A_2$	$\tilde{a}_{3,7}$	none	$\bar{a}_{1,2}, a_{1,9}^\epsilon$
$a_{2,17}^d$ $d > 0$ [ $d \neq 0$ ]	$F + dL_3; P_0$	$A_2$	$a_{3,5}$	none	$a_{1,3}, a_{1,7}^d$
$a_{2,18}^d$ $0 <  d  < 1$ [ $d \neq 0, \pm 1$ ]	$F + dK_2; P_0 - P_2$	$A_2$	$a_{3,4}$	none	$a_{1,8}^d, \bar{a}_{1,11}$
$a_{2,19}^\epsilon$ $\epsilon = \pm 1$ * [ $\epsilon = \pm 1$ ]	$F - K_2 + \epsilon(P_0 - P_2); P_0 + P_2$	$A_2$	$a_{3,6}$	none	$a_{1,9}^\epsilon, \bar{a}_{1,11}$
$a_{2,20}^\epsilon$ $\epsilon = \pm 1$ [ $\epsilon = \pm 1$ ]	$F + \frac{1}{2}K_2; -K_1 + L_3 + \epsilon(P_0 + P_2)$	$A_2$	self	none	$a_{1,5}^\epsilon, a_{1,8}^{1/2}$
$\bar{a}_{2,21} = b_{2,4}$	$F + K_2; P_0 - P_2$	$A_2$	$b_{5,1}$	none	$\bar{a}_{1,11}, \bar{a}_{1,12}$
$a_{1,1}$	$F$	$A_1$	$a_{4,4}$	$F$	
$\tilde{a}_{1,1}$	$K_2$	$A_1$	$\sim a_{4,4}$	$K_2$	
$a_{1,2}$	$P_1$	$A_1$	$a_{5,6}$	$P_1$	
$\tilde{a}_{1,2}$	$-K_1 + L_3$	$A_1$	$\sim a_{5,6}$	$-K_1 + L_3$	
$a_{1,3}$	$P_0$	$A_1$	$a_{5,7}$	$P_0$	
$a_{1,4}^\epsilon$ $\epsilon = \pm 1$ [ $\epsilon = \pm 1$ ]	$F + \epsilon(-K_1 + L_3)$	$A_1$	$a_{2,1}$	$F + \epsilon(-K_1 + L_3)$	
$\tilde{a}_{1,4}^\epsilon$ $\epsilon = \pm 1$ [ $\epsilon = \pm 1$ ]	$K_2 - \epsilon P_1$	$A_1$	$\tilde{a}_{2,1}$	$K_2 - \epsilon P_1$	
$a_{1,5}^\epsilon$ $\epsilon = \pm 1$ * [ $\epsilon = \pm 1$ ]	$-K_1 + L_3 + \epsilon(P_0 + P_2)$	$A_1$	$a_{5,18}^\epsilon$	$-K_1 + L_3 + \epsilon(P_0 + P_2)$	
$a_{1,6}^\epsilon$ $\epsilon = \pm 1$ [ $\epsilon = \pm 1$ ]	$L_3 + \epsilon P_0$	$A_1$	$a_{2,3}$	$L_3 + \epsilon P_0$	
$a_{1,7}^\epsilon$ $e > 0$ [ $e \neq 0$ ]	$F + eL_3$	$A_1$	$a_{2,7}$	$F + eL_3$	
$a_{1,8}^\epsilon$ $0 < e < 1$ [ $e > 0$ ]	$F + eK_2$	$A_1$	$a_{2,6}$	$F + eK_2$	
$a_{1,9}^\epsilon$ $\epsilon = \pm 1$ [ $\epsilon = \pm 1$ ]	$F - K_2 + \epsilon(P_0 - P_2)$	$A_1$	$a_{2,8}$	$F - K_2 + \epsilon(P_0 - P_2)$	
$\bar{a}_{1,10} = d_{1,1}$	$L_3$	$A_1$	$d_{4,1}$	$L_3$	
$\bar{a}_{1,11} = b_{1,4}$	$P_0 - P_2$	$A_1$	$b_{7,1}$	$P_0 - P_2$	
$\bar{a}_{1,12} = b_{1,5}$	$F + K_2$	$A_1$	$b_{4,2}$	$F + K_2$	



TABLE V. Subalgebras of the optical algebra LOpt(2,1).

Name and range of parameters	Generators	Isomorphism class	Normalizer	Invariants	Maximal subalgebras
$b_{7,1}$	$W; K_1, K_2, L_3, M, Q, N$	LOpt(2,1)	self	$\frac{1}{N}[MQK_1 + \frac{1}{2}(K_2 - L_3)M^2 - \frac{1}{2}(K_2 + L_3)Q^2]_{\text{sym}} + K_1^2 + K_2^2 - L_3^2$	$b_{6,1}, \bar{b}_{6,2}, b_{5,1}, b_{5,2}$
$b_{6,1}$	$; K_1, K_2, L_3, M, Q, N$	LSch <sub>1</sub>	$b_{7,1}$	$[MQK_1 + \frac{1}{2}(K_2 - L_3)M^2 - \frac{1}{2}(K_2 + L_3)Q^2]_{\text{sym}} + N(K_1^2 + K_2^2 - L_3^2), N$	$\bar{b}_{5,4}, b_{4,1}, b_{4,3}$
$\bar{b}_{6,2} = a_{6,2}$	$W, K_1; K_2 + L_3, M, Q, N$		self	none	$\bar{b}_{5,3}, \bar{b}_{5,4}, \bar{b}_{5,5}, \bar{b}_{5,6}^a, \bar{b}_{5,7}, \bar{b}_{5,8}, \bar{b}_{5,9}$
$b_{5,1}$	$\{; K_1, K_2, L_3\} \oplus \{W; N\}$	$A_{3,8} \oplus A_2$	self	$K_1^2 + K_2^2 - L_3^2$	$b_{4,1}, b_{4,2}, \bar{b}_{4,9}, b_{3,1}$
$b_{5,2}$	$W, L_3; M, Q, N$	$A_{5,37}$	self	$(Q^2 + M^2 + 4NL_3)/N$	$b_{4,3}, b_{4,4}^b, b_{4,5}, b_{3,1}$
$\bar{b}_{5,3} = a_{5,1}$	$W + K_1; K_2 + L_3, M, Q, N$	$A_{5,30}^0$	$\bar{b}_{6,2}$	$M^2 + 2N(K_2 + L_3)$	$\bar{b}_{4,6}, \bar{b}_{4,10}, \bar{b}_{4,16}$
$\bar{b}_{5,4} = a_{5,2}$	$K_1; K_2 + L_3, M, Q, N$	$A_{5,30}^{-1}$	$\bar{b}_{6,2}$	$N$	$\bar{b}_{4,7}, \bar{b}_{4,10}, \bar{b}_{4,15}$
$\bar{b}_{5,5} = a_{5,3}$	$W, K_2 + L_3; M, Q, N$	$A_{5,30}^1$	$\bar{b}_{6,2}$	$2(K_2 + L_3) + M^2/N$	$b_{4,5}, \tilde{b}_{4,7}, \bar{b}_{4,10}, \bar{b}_{4,14}$
$\bar{b}_{5,6}^a = a_{5,4}^{(1+a)/(1-a)}$ $a \neq 0, +1$	$W + aK_1; K_2 + L_3, M, Q, N$	$A_{5,30}^{(1-a)/(1+a)}$	$\bar{b}_{6,2}$	$[M^2 + 2N(K_2 + L_3)]N^{a-1}$	$\bar{b}_{4,10}, \bar{b}_{4,13}^a, \bar{b}_{4,17}^a, \bar{b}_{4,18} (a = -\frac{1}{3})$
$\bar{b}_{5,7} = a_{5,5}$	$W - K_1; K_2 + L_3, M, Q, N$	$A_{5,32}^0$	$\bar{b}_{6,2}$	$[M^2 + 2N(K_2 + L_3)]/N^2$	$\bar{b}_{4,10}, \bar{b}_{4,11}, \bar{b}_{4,12}, \bar{b}_{4,16}$
$\bar{b}_{5,8} = a_{5,6}$	$W, K_1; K_2 + L_3, M, N$	$A_{5,33}^{2/2}$	self	$(K_2 - L_3)N/M^2$	$\bar{b}_{4,6}, \bar{b}_{4,7}, \tilde{b}_{4,7}, \bar{b}_{4,8}, \tilde{\bar{b}}_{4,8}, \bar{b}_{4,9}, \bar{b}_{4,12}, \bar{b}_{4,13}$
$\bar{b}_{5,9} = a_{5,8}$	$W, K_1; M, Q, N$	$A_{5,36}$	self	$[QM + MQ - 2K_1(W + 2K_1)]/N$	$b_{4,5}, \bar{b}_{4,8}, \bar{b}_{4,15}, \bar{b}_{4,16}, \bar{b}_{4,17}^b$
$b_{4,1}$	$N \oplus \{; K_1, K_2, L_3\}$	$A_1 \oplus A_{3,8}$	$b_{5,1}$	$N, K_1^2 + K_2^2 - L_3^2$	$\bar{b}_{3,3}, \bar{b}_{3,8}, b_{2,1}$
$b_{4,2}$	$W \oplus \{; K_1, K_2, L_3\}$	$A_1 \oplus A_{3,8}$	self	$W, K_1^2 + K_2^2 - L_3^2$	$\bar{b}_{3,3}, \bar{b}_{3,7}, b_{2,2}$
$b_{4,3}$	$L_3; Q, M, N$	$A_{4,10}$	$b_{5,2}$	$N, M^2 + Q^2 + 4L_3N$	$b_{3,2}, b_{2,1}$
$b_{4,4}^b$ $b > 0 * [b \neq 0]$	$W + bL_3; Q, M, N$	$A_{4,11}^b$	$b_{5,2}$	none	$b_{3,2}, b_{3,3}^b$
$b_{4,5} = a_{4,18}$	$W; Q, M, N$	$A_{4,9}^1$	$b_{7,1}$	none	$b_{3,2}, \bar{b}_{3,19}$
$\bar{b}_{4,6} = a_{4,1}$	$M \oplus \{W + K_1; K_2 + L_3, N\}$	$A_1 \oplus A_{3,4}$	$\bar{b}_{5,8}$	$M, N(K_2 + L_3)$	$\bar{b}_{3,4}, \bar{b}_{3,5}, \tilde{b}_{3,5}, \bar{b}_{3,16}, \bar{b}_{3,17}$
$\bar{b}_{4,7} = a_{4,2}$	$N \oplus \{K_1; K_2 + L_3, M\}$	$A_1 \oplus A_{3,5}^2$	$\bar{b}_{5,8}$	$N, (K_2 + L_3)/M^2$	$\bar{b}_{3,4}, \bar{b}_{3,8}, \bar{b}_{3,9}, \tilde{\bar{b}}_{3,19}, \tilde{b}_{3,20}$
$\tilde{b}_{4,7} \sim a_{4,2}$	$K_2 + L_3 \oplus \{W; M, N\}$	$A_1 \oplus A_{3,5}^2$	$\bar{b}_{5,8}$	$K_2 + L_3, M^2/N$	$\bar{b}_{3,4}, \tilde{b}_{3,8}, \tilde{b}_{3,9}, \bar{b}_{3,13}, \bar{b}_{3,20}$

TABLE V. (continued)

Name and range of parameters	Generators	Isomorphism class	Normalizer	Invariants	Maximal subalgebras
$\bar{b}_{4,8} = a_{4,5}$	$\{W + K_1; N\}$ $\oplus \{K_1; M\}$	$A_2 \oplus A_2$	self	none	$\bar{b}_{3,5}, \bar{b}_{3,6}, \bar{b}_{3,7},$ $\bar{b}_{3,9}, \bar{b}_{3,12},$ $\bar{b}_{3,18}, \bar{b}_{3,19}, \bar{b}_{3,22}$
$\tilde{b}_{4,8} \sim a_{4,5}$	$\{W + K_1; K_2 + L_3\}$ $\oplus \{W; M\}$	$A_2 \oplus A_2$	self	none	$\tilde{b}_{3,5}, \tilde{b}_{3,6}, \tilde{b}_{3,7}, \tilde{b}_{3,9}, \tilde{b}_{3,12},$ $\tilde{b}_{3,18}, \tilde{b}_{3,19}, \tilde{b}_{3,22}$
$\bar{b}_{4,9} = a_{4,6}$	$\{K_1; K_2 + L_3\}$ $\oplus \{W; N\}$	$A_2 \oplus A_2$	self	none	$\bar{b}_{3,7}, \tilde{b}_{3,7}, \bar{b}_{3,8},$ $\tilde{b}_{3,8}, \bar{b}_{3,15}, \bar{b}_{3,16}, \bar{b}_{3,22}$
$\bar{b}_{4,10} = a_{4,7}$ $\bar{b}_{4,11} \sim a_{4,8}^{-1}$	$K_2 + L_3, Q; M, N$ $W - K_1 + Q;$ $K_2 + L_3, M, N$	$A_{4,1}$ $A_{4,4}$	$\bar{b}_{6,2}$ $\bar{b}_{5,7}$	$N, M^2 + 2N(K_2 + L_3)$ $N \exp(-M/N),$ $(2N(K_2 + L_3) + M^2)/N^2$	$b_{3,2}, \bar{b}_{3,4}, \bar{b}_{3,10}$ $\bar{b}_{3,4}, \bar{b}_{3,11}$
$\bar{b}_{4,12} = a_{4,9}$	$W - K_1;$ $K_2 + L_3, M, N$	$A_{4,5}^{1;1}$	$a_{7,1}$	$M/N, (K_2 + L_3)/N$	$\bar{b}_{3,4}, \bar{b}_{3,12}, \tilde{b}_{3,12},$ $\bar{b}_{3,13}, \bar{b}_{3,14}, \bar{b}_{3,15}$
$\bar{b}_{4,13} = a_{4,10}^{(b-1)/(b+1)}$ $0 <  b  < 1$ $[b \neq 0, \pm 1]$	$W - bK_1;$ $K_2 + L_3, M, N$	$A_{4,5}^{b(b+1)/2}$	$b_{5,8}$	$N^{(1+b)/2}/M, N^b/(K_2 + L_3)$	$\bar{b}_{3,4}, \bar{b}_{3,18} (b = -3), \bar{b}_{3,23}$ $\tilde{b}_{3,18} (b = -\frac{1}{3}), \bar{b}_{3,22}, \tilde{b}_{3,22}$
$\bar{b}_{4,14} = a_{4,12}^{\epsilon}$ $\epsilon = 1 * [\epsilon = \pm 1]$	$W + \epsilon(K_2 + L_3);$ $M, Q, N$	$A_{4,7}$	$\bar{b}_{5,3}$	none	$b_{3,2}, \bar{b}_{3,20}$
$\bar{b}_{4,15} = a_{4,13}$	$K_1; M, Q, N$	$A_{4,8}$	$\bar{b}_{5,9}$	$N, QM + MQ + 4NK_1$	$b_{3,2}, \bar{b}_{3,9}$
$\bar{b}_{4,16} = a_{4,14}$	$W + K_1, M; Q, N$	$A_{4,9}^0$	$\bar{b}_{5,9}$	none	$b_{3,2}, \bar{b}_{3,5}, \bar{b}_{3,11},$ $\bar{b}_{3,12}$
$\bar{b}_{4,17} = a_{4,15}^{(1-b)/(1+b)}$ $b > 0, b \neq 1$	$W - bK_1; M, Q, N$	$A_{4,9}^{(1-b)/(1+b)}$	$\bar{b}_{5,9}$	none	$b_{3,2}, \bar{b}_{3,18} (b = 3), \tilde{b}_{3,22}$
$\bar{b}_{4,18} \sim a_{4,16}^{-1}$	$W - \frac{1}{3}K_1;$ $K_2 + L_3 + Q, M, N$	$A_{4,9}^{1/3}$	$\bar{b}_{5,9}^{1/3}$	none	$\bar{b}_{3,10}, \bar{b}_{3,21}, \tilde{b}_{3,22}^{1/3}$
$b_{3,1}$	$L_3 \oplus \{W; N\}$	$A_1 \oplus A_2$	self	$L_3$	$b_{2,1}, b_{2,2}, b_{2,3}, b_{2,4}$
$b_{3,2} = \bar{a}_{3,25}$	$Q, M; N$	$A_{3,1}$	$b_{7,1}$	$N$	$\tilde{b}_{2,6}$
$\bar{b}_{3,3} = e_{3,1}$	$; K_1, K_2, L_3$	$A_{3,8}$	$e_{6,1}$	$K_1^2 + K_2^2 - L_3^2$	$\tilde{b}_{2,4}, \bar{b}_{1,6}$
$\bar{b}_{3,4} = a_{3,1}$	$K_2 + L_3, M, N$	$3A_1$	$a_{7,1}$	$K_2 + L_3, M, N$	$\bar{b}_{2,6}, \tilde{b}_{2,6}, \bar{b}_{2,7},$ $\bar{b}_{2,8}, \tilde{b}_{2,8}$
$\bar{b}_{3,5} = a_{3,2}$	$M \oplus \{W + K_1; N\}$	$A_1 \oplus A_2$	$\bar{b}_{4,8}$	$M$	$\bar{b}_{2,5}, \tilde{b}_{2,5}, \bar{b}_{2,13}, \tilde{b}_{2,14}$
$\tilde{b}_{3,5} \sim a_{3,2}$	$M \oplus \{W + K_1;$ $K_2 + L_3\}$	$A_1 \oplus A_2$	$\tilde{b}_{4,8}$	$M$	$\bar{b}_{2,5}, \bar{b}_{2,6}, \tilde{b}_{2,13}, \bar{b}_{2,14}$

TABLE V. (continued)

Name and range of parameters	Generators	Isomorphism class	Normalizer	Invariants	Maximal subalgebras
$\bar{b}_{3,6} \sim a_{3,3}$	$W + K_1 \oplus \{W - K_1; M\}$	$A_1 \oplus A_2$	self	$W + K_1$	$\bar{b}_{2,5}, \bar{b}_{2,9}, \bar{b}_{2,12}, \bar{b}_{2,16}, \bar{b}_{2,17}, \tilde{b}_{2,17}$
$\bar{b}_{3,7} \sim a_{3,4}$	$W \oplus \{K_1; K_2 + L_3\}$	$A_1 \oplus A_2$	self	$W$	$\tilde{b}_{2,4}, \bar{b}_{2,9}, \tilde{b}_{2,10}, \tilde{b}_{2,13}, \bar{b}_{2,13}, \tilde{b}_{2,19}$
$\tilde{b}_{3,7} = a_{3,4}$	$K_1 \oplus \{W; N\}$	$A_1 \oplus A_2$	self	$K_1$	$b_{2,4}, \bar{b}_{2,9}, \bar{b}_{2,10}, \bar{b}_{2,13}, \bar{b}_{2,19}$
$\bar{b}_{3,8} = a_{3,6}$	$N \oplus \{K_1; K_2 + L_3\}$	$A_1 \oplus A_2$	$\bar{b}_{4,9}$	$N$	$\tilde{b}_{2,4}, \tilde{b}_{2,8}, \bar{b}_{2,10}, \bar{b}_{2,20}$
$\tilde{b}_{3,8} \sim a_{3,6}$	$K_2 + L_3 \oplus \{W; N\}$	$A_1 \oplus A_2$	$\bar{b}_{4,9}$	$K_2 + L_3$	$b_{2,4}, \tilde{b}_{2,8}, \tilde{b}_{2,10}, \tilde{b}_{2,20}$
$\bar{b}_{3,9} = a_{3,7}$	$N \oplus \{K_1; M\}$	$A_1 \oplus A_2$	$\bar{b}_{4,8}$	$N$	$\tilde{b}_{2,6}, \bar{b}_{2,10}, \bar{b}_{2,17}, \bar{b}_{2,18}$
$\tilde{b}_{3,9} \sim a_{3,7}$	$K_2 + L_3 \oplus \{W; M\}$	$A_1 \oplus A_2$	$\tilde{b}_{4,8}$	$K_2 + L_3$	$\bar{b}_{2,6}, \tilde{b}_{2,10}, \tilde{b}_{2,17}, \tilde{b}_{2,18}$
$\bar{b}_{3,10} \sim a_{3,8}^{-1}$	$K_2 + L_3 + Q, M; N$	$A_{3,1}$	$\bar{b}_{5,6}^{-1/3}$	$N$	$\tilde{b}_{2,6}, \bar{b}_{2,11}$
$\bar{b}_{3,11} \sim a_{3,9}^{-1}$	$W + K_1 + M; Q, N$	$A_{3,2}$	$\bar{b}_{4,17}$	$N \exp(Q/N)$	$\tilde{b}_{2,6}, \tilde{b}_{2,14}$
$\bar{b}_{3,12} = \tilde{a}_{3,10}$	$W - K_1; M, N$	$A_{3,3}$	$\bar{b}_{5,9}$	$M/N$	$\tilde{b}_{2,6}, \bar{b}_{2,12}, \bar{b}_{2,13}$
$\tilde{b}_{3,12} \sim \tilde{a}_{3,10}$	$W - K_1; K_2 + L_3, M$	$A_{3,3}$	$\sim \bar{b}_{5,9}$	$(K_2 + L_3)/M$	$\bar{b}_{2,6}, \bar{b}_{2,12}, \tilde{b}_{2,12}, \tilde{b}_{2,13}$
$\bar{b}_{3,13} = a_{3,11}$	$W - K_1; K_2 + L_3 + N, M$	$A_{3,3}$	$a_{4,17}$	$(K_2 + L_3 + N)/M$	$\bar{b}_{2,7}, \bar{b}_{2,12}, \tilde{b}_{2,12}$
$\bar{b}_{3,14} \sim a_{3,12}$	$W - K_1; K_2 + L_3 - N, M$	$A_{3,3}$	$\sim a_{4,6}$	$(K_2 + L_3 - N)/M$	$\bar{b}_{2,8}, \bar{b}_{2,12}, \tilde{b}_{2,12}, \tilde{b}_{2,13}, \bar{b}_{2,15}$
$\bar{b}_{3,15} = a_{3,12}$	$W - K_1; K_2 + L_3, N$	$A_{3,3}$	$\bar{b}_{4,9}$	$(K_2 + L_3)/N$	$\tilde{b}_{2,8}, \bar{b}_{2,12}, \bar{b}_{2,13}, \tilde{b}_{2,13}, \bar{b}_{2,15}$
$\bar{b}_{3,16} = a_{3,13}$	$W + K_1; K_2 + L_3, N$	$A_{3,4}$	$\bar{b}_{5,8}$	$N(K_2 + L_3)$	$\tilde{b}_{2,8}, \bar{b}_{2,13}, \tilde{b}_{2,13}$
$\bar{b}_{3,17} \sim a_{3,14}^{-1}$	$W + K_1 + M; K_2 + L_3, N$	$A_{3,4}$	$\bar{b}_{4,6}$	$N(K_2 + L_3)$	$\tilde{b}_{2,8}, \bar{b}_{2,14}, \tilde{b}_{2,14}$
$\bar{b}_{3,18} = a_{3,15}$	$W + 3K_1; M, N$	$A_{3,4}$	$\bar{b}_{4,8}$	$NM$	$\tilde{b}_{2,6}, \bar{b}_{2,16}, \bar{b}_{2,19}$
$\tilde{b}_{3,18} \sim a_{3,15}$	$W + \frac{1}{3}K_1; K_2 + L_3, M$	$A_{3,4}$	$\tilde{b}_{4,8}$	$M(K_2 + L_3)$	$\bar{b}_{2,6}, \bar{b}_{2,16}^{-1/3}, \tilde{b}_{2,19}^{-1/3}$
$\bar{b}_{3,19} = a_{3,16}$	$W; M, N$	$A_{3,5}^{1/2}$	$\bar{b}_{5,8}$	$M^2/N$	$b_{2,4}, \tilde{b}_{2,6}, \tilde{b}_{2,17}$
$\tilde{b}_{3,19} \sim a_{3,16}$	$K_1; K_2 + L_3, M$	$A_{3,5}^{1/2}$	$\bar{b}_{5,8}$	$M^2/(K_2 + L_3)$	$\tilde{b}_{2,4}, \bar{b}_{2,6}, \bar{b}_{2,17}$
$\bar{b}_{3,20} = a_{3,17}^\epsilon$ $\epsilon = 1 * [\epsilon = \pm 1]$	$W + \epsilon(K_2 + L_3); M, N$	$A_{3,5}^{1/2}$	$\tilde{b}_{4,7}$	$M^2/N$	$\tilde{b}_{2,6}, \tilde{b}_{2,18}, \tilde{b}_{2,20}$
$\tilde{b}_{3,20} \sim a_{3,17}^\epsilon$ $\epsilon = 1 * [\epsilon = \pm 1]$	$K_1 + \epsilon N; K_2 + L_3, M$	$A_{3,5}^{1/2}$	$\bar{b}_{4,7}$	$M^2/(K_2 + L_3)$	$\bar{b}_{2,6}, \bar{b}_{2,18}, \bar{b}_{2,20}$
$\bar{b}_{3,21} \sim a_{3,18}^{-1}$	$W - \frac{1}{3}K_1; K_2 + L_3 + Q, N$	$A_{3,5}^{1/2}$	self	$(K_2 + L_3 + Q)^3/N$	$\bar{b}_{2,11}, \bar{b}_{2,19}^{1/3}, \bar{b}_{2,21}$

TABLE V. (continued)

Name and range of parameters	Generators	Isomorphism class	Normalizer	Invariants	Maximal subalgebras
$\bar{b}_{3,22}^c \sim a_{3,19}^{(c+1)/(c-1)}$ $c \neq 0, \frac{1}{3}, \pm 1$	$W + cK_1; K_2 + L_3, M$	$A_{3,5}^h$ $h = \frac{2c}{c-1}, -1 < c < \frac{1}{3}$ $h = \frac{c-1}{2c}, c > \frac{1}{3}$ or $c < -1$	$\bar{b}_{4,8}$	$(K_2 + L_3)M^{2c/(1-c)}$	$\bar{b}_{2,6}, \bar{b}_{2,16}^c, \bar{b}_{2,19}^c$
$\bar{b}_{3,22}^c = a_{3,19}^{(1-c)/(1+c)}$ $c \neq 0, \pm 1, -3$	$W - cK_1; M, N$	$A_{3,5}^h$ $h = \frac{c+1}{2}, -3 < c < 1$ $h = \frac{2}{c+1}, c > 1$ or $c < -3$	$\bar{b}_{4,8}$	$MN^{-(c+1)/2}$	$\bar{b}_{2,6}, \bar{b}_{2,16}^c, \bar{b}_{2,19}^c$
$\bar{b}_{3,23}^c = a_{3,20}^{(1-c)/(1+c)}$ $0 <  c  < 1$ [ $c \neq 0, \pm 1$ ]	$W - cK_1; K_2 + L_3, N$	$A_{3,5}$	$\bar{b}_{4,9}$	$(K_2 + L_3)N^{-c}$	$\bar{b}_{2,6}, \bar{b}_{2,19}^c, \bar{b}_{2,19}^c$
$b_{2,1}$	$L_3, N$	$2A_1$	$b_{3,1}$	$L_3, N$	$b_{1,1}, b_{1,2}, b_{1,4}, \bar{b}_{1,6}$
$b_{2,2}$	$W, L_3$	$2A_1$	self	$W, L_3$	$b_{1,3}, b_{1,5}, \bar{b}_{1,6}$
$b_{2,3}^d$ $d > 0$ [ $d \neq 0$ ]	$W + dL_3; N$	$A_2$	$b_{3,1}$	none	$b_{1,3}, b_{1,4}$
$b_{2,4} = \bar{a}_{2,21}$	$W; N$	$A_2$	$b_{5,1}$	none	$b_{1,4}, b_{1,5}$
$\bar{b}_{2,4} \sim \bar{a}_{2,21}$	$K_1; K_2 + L_3$	$A_2$	$\sim b_{5,1}$	none	$\bar{b}_{1,4}, \bar{b}_{1,5}$
$\bar{b}_{2,5} = \bar{a}_{2,1}$	$W + K_1, M$	$2A_1$	$\bar{b}_{3,6}$	$W + K_1, M$	$\bar{b}_{1,7}, \bar{b}_{1,8}, \bar{b}_{1,10}$
$\bar{b}_{2,6} \sim a_{2,2}$	$K_2 + L_3, M$	$2A_1$	$\sim \bar{b}_{6,2}$	$K_2 + L_3, M$	$\bar{b}_{1,4}, \bar{b}_{1,8}, \bar{b}_{1,8}$
$\bar{b}_{2,6} = a_{2,2}$	$M, N$	$2A_1$	$\bar{b}_{6,2}$	$M, N$	$b_{1,4}, \bar{b}_{1,8}$
$\bar{b}_{2,7} = a_{2,4}$	$K_2 + L_3 + N, M$	$2A_1$	$a_{5,7}$	$K_2 + L_3 + N, M$	$\bar{b}_{1,8}, \bar{b}_{1,8}$
$\bar{b}_{2,8} \sim a_{2,5}$	$K_2 + L_3 - N, M$	$2A_1$	$\sim \bar{b}_{5,8}$	$K_2 + L_3 - N, M$	$\bar{b}_{1,4}, \bar{b}_{1,8}, \bar{b}_{1,8}, \bar{b}_{1,9}$
$\bar{b}_{2,8} = a_{2,5}$	$K_2 + L_3, N$	$2A_1$	$\bar{b}_{5,8}$	$K_2 + L_3, N$	$b_{1,4}, \bar{b}_{1,4}, \bar{b}_{1,8}, \bar{b}_{1,9}$
$\bar{b}_{2,9} = a_{2,6}$	$W, K_1$	$2A_1$	self	$W, K_1$	$b_{1,5}, \bar{b}_{1,5}, \bar{b}_{1,7}, \bar{b}_{1,12}$
$\bar{b}_{2,10} = a_{2,8}$	$K_1, N$	$2A_1$	$\bar{b}_{3,7}$	$K_1, N$	$b_{1,4}, \bar{b}_{1,5}, \bar{b}_{1,13}$
$\bar{b}_{2,10} \sim a_{2,8}$	$W, K_2 + L_3$	$2A_1$	$\bar{b}_{3,7}$	$W, K_2 + L_3$	$\bar{b}_{1,4}, b_{1,5}, \bar{b}_{1,13}$
$\bar{b}_{2,11} \sim a_{2,9}^1$	$K_2 + L_3 + Q, N$	$2A_1$	$\bar{b}_{3,20}$	$K_2 + L_3 + Q, N$	$b_{1,4}, \bar{b}_{1,11}$
$\bar{b}_{2,12} = \bar{a}_{2,10}$	$W - K_1; M$	$A_2$	$\bar{b}_{3,6}$	none	$\bar{b}_{1,7}, \bar{b}_{1,8}$
$\bar{b}_{2,12} \sim \bar{a}_{2,10}$	$W - K_1; K_2 + L_3 + N$	$A_2$	$\sim \bar{b}_{3,6}$	none	$\bar{b}_{1,7}, \bar{b}_{1,8}$
$\bar{b}_{2,13} = a_{2,11}$	$W + K_1; N$	$A_2$	$\bar{b}_{4,8}$	none	$b_{1,4}, \bar{b}_{1,7}$
$\bar{b}_{2,13} \sim \bar{a}_{2,11}$	$W - K_1; K_2 + L_3$	$A_2$	$\bar{b}_{4,8}$	none	$\bar{b}_{1,4}, \bar{b}_{1,7}$

TABLE V. (continued)

Name and range of parameters	Generators	Isomorphism class	Normalizer	Invariants	Maximal subalgebras
$\tilde{b}_{2,13} \sim a_{2,11}$	$W + K_1; K_2 + L_3$	$A_2$	$\tilde{b}_{4,8}$	none	$\tilde{b}_{1,4}, \bar{b}_{1,7}$
$\bar{b}_{2,14} \sim a_{2,12}^1$	$W + K_1 + M;$ $K_2 + L_3$	$A_2$	$\tilde{b}_{3,5}$	none	$\tilde{b}_{1,4}, \bar{b}_{1,10}$
$\tilde{b}_{2,14} \sim a_{2,12}^{-1}$	$W + K_1 + M; N$	$A_2$	$\bar{b}_{3,5}$	none	$b_{1,4}, \bar{b}_{1,10}$
$\bar{b}_{2,15} = a_{2,13}$	$W - K_1;$ $K_2 + L_3 - N$	$A_2$	$\sim a_{3,5}$	none	$\bar{b}_{1,7}, \bar{b}_{1,9}$
$\bar{b}_{2,16}^d = \tilde{a}_{2,14}^{d-1/(1+d)}$ $0 <  d  < 1 [d \neq 0, \pm 1]$	$W - dK_1; M$	$A_2$	$\bar{b}_{3,6}$	none	$\bar{b}_{1,8}, \bar{b}_{1,12}^d$
$\bar{b}_{2,17} = a_{2,15}$	$K_1; M$	$A_2$	$\bar{b}_{4,8}$	none	$\tilde{b}_{1,5}, \bar{b}_{1,8}$
$\tilde{b}_{2,17} \sim a_{2,15}$	$W; M$	$A_2$	$\bar{b}_{4,8}$	none	$b_{1,5}, \bar{b}_{1,8}$
$\bar{b}_{2,18}^\epsilon = a_{2,16}^\epsilon$ $\epsilon = 1 * [\epsilon = \pm 1]$	$K_1 + \epsilon N; M$	$A_2$	$\bar{b}_{3,9}$	none	$\bar{b}_{1,8}, \bar{b}_{1,13}$
$\tilde{b}_{2,18}^\epsilon \sim \bar{a}_{2,16}^\epsilon$ $\epsilon = 1 * [\epsilon = \pm 1]$	$W - \epsilon(K_2 + L_3); M$	$A_2$	$\tilde{b}_{3,9}$	none	$\bar{b}_{1,8}, \tilde{b}_{1,13}$
$\bar{b}_{2,19} \sim a_{2,18}^{(1-d)/(1+d)}$ $d > 0, \neq 1$	$W - dK_1; N$	$A_2$	$\tilde{b}_{3,7}$	none	$b_{1,4}, \bar{b}_{1,12}^d$
$\tilde{b}_{2,19} \sim a_{2,18}^{(1-d)/(1+d)}$ $d > 0, \neq 1 [d \neq 0, \pm 1]$	$W - dK_1; K_2 + L_3$	$A_2$	$\bar{b}_{3,7}$	none	$\tilde{b}_{1,4}, \bar{b}_{1,12}^d$
$\bar{b}_{2,20}^\epsilon = a_{2,19}^\epsilon$ $\epsilon = 1 * [\epsilon = \pm 1]$	$K_1 + \epsilon N; K_2 + L_3$	$A_2$	$\bar{b}_{3,8}$	none	$\tilde{b}_{1,4}, \bar{b}_{1,13}$
$\tilde{b}_{2,20}^\epsilon \sim a_{2,19}^\epsilon$ $\epsilon = 1 * [\epsilon = \pm 1]$	$W + \epsilon(K_2 + L_3); N$	$A_2$	$\tilde{b}_{3,8}$	none	$b_{1,4}, \tilde{b}_{1,13}$
$\bar{b}_{2,21} = a_{2,20}^{-1}$	$W - \frac{1}{3}K_1;$ $K_2 + L_3 + Q$	$A_2$	self	none	$\bar{b}_{1,11}, \bar{b}_{1,12}^{1/3}$
$b_{1,1}$	$L_3 + N$	$A_1$	$b_{2,1}$	$L_3 + N$	
$b_{1,2}$	$L_3 - N$	$A_1$	$b_{2,1}$	$L_3 - N$	
$b_{1,3}^e, e > 0 [e \neq 0]$	$W + eL_3$	$A_1$	$b_{2,2}$	$W + eL_3$	
$b_{1,4} = \bar{a}_{1,11}$	$N$	$A_1$	$b_{7,1}$	$N$	
$\tilde{b}_{1,4} \sim \bar{a}_{1,11}$	$K_2 + L_3$	$A_1$	$\sim b_{7,1}$	$K_2 + L_3$	
$b_{1,5} = \bar{a}_{1,12}$	$W$	$A_1$	$b_{4,2}$	$W$	
$\tilde{b}_{1,5} \sim \bar{a}_{1,12}$	$K_1$	$A_1$	$\sim b_{4,2}$	$K_1$	
$\bar{b}_{1,6} = e_{1,1}$	$L_3$	$A_1$	$e_{4,1}$	$L_3$	
$\bar{b}_{1,7} = a_{1,1}$	$W - K_1$	$A_1$	$a_{4,4}$	$W - K_1$	
$\bar{b}_{1,8} = a_{1,2}$	$M$	$A_1$	$\bar{b}_{5,8}$	$M$	
$\tilde{b}_{1,8} \sim a_{1,2}$	$K_2 + L_3 + N$	$A_1$	$\sim \bar{b}_{5,8}$	$K_2 + L_3 + N$	
$\bar{b}_{1,9} = a_{1,3}$	$K_2 + L_3 - N$	$A_1$	$a_{5,7}$	$K_2 + L_3 - N$	
$\bar{b}_{1,10} \sim \bar{a}_{1,4}^{-1}$	$W + K_1 + M$	$A_1$	$\bar{b}_{2,5}$	$W + K_1 + M$	
$\bar{b}_{1,11} \sim \bar{a}_{1,5}^{-1}$	$K_2 + L_3 + Q$	$A_1$	$\bar{b}_{3,20}$	$K_2 + L_3 + Q$	
$\bar{b}_{1,12}^e = a_{1,8}^{(1-e)/(1+e)}$ $0 < e < 1 [e > 0, \neq 1]$	$W - eK_1$	$A_1$	$\bar{b}_{2,9}$	$W - eK_1$	
$\bar{b}_{1,13} = a_{1,9}^1$	$K_1 + N$	$A_1$	$\bar{b}_{2,10}$	$K_1 + N$	
$\tilde{b}_{1,13} = a_{1,9}^1$ $\epsilon = 1 [\epsilon = \pm 1]$	$W + \epsilon(K_2 + L_3)$	$A_1$	$\tilde{b}_{2,10}$	$W + \epsilon(K_2 + L_3)$	

TABLE VI. Subalgebras of  $LO(3) \oplus LO(2)$ .

Name and range of parameters	Generators	Isomorphism class	Normalizer	Invariants	Maximal subalgebras
$c_{4,1}$	$D \oplus \{A, B, C\}$	$A_1 \oplus A_{3,9}$	self	$D, A^2 + B^2 + C^2$	$c_{3,1}, c_{2,1}$
$c_{3,1}$	$\{A, B, C\}$	$A_{3,9}$	$c_{4,1}$	$A^2 + B^2 + C^2$	$\bar{c}_{1,3}$
$c_{2,1}$	$D, C$	$2A_1$	self	$D, C$	$c_{1,1}, c_{1,2}, \bar{c}_{1,3}, \bar{c}_{1,4}$
$c_{1,1}$	$D$	$A_1$	$c_{4,1}$	$D$	
$c_{1,2}^e$ $e > 0, \neq 1$	$D + eC$	$A_1$	$c_{2,1}$	$D + eC$	
$\bar{c}_{1,3} = d_{1,1}$	$C$	$A_1$	$d_{4,1}$	$C$	
$\bar{c}_{1,4} = e_{1,1}$	$A + D$	$A_1$	$e_{4,1}$	$A + D$	

TABLE VII. Subalgebras of  $LO(2) \oplus LO(2,1)$ .

Name and range of parameters	Generators	Isomorphism class	Normalizer	Invariants	Maximal subalgebras
$d_{4,1}$	$C \oplus \{D, E, F\}$	$A_1 \oplus A_{3,8}$	self	$C, D^2 - E^2 - F^2$	$d_{3,1}, \bar{d}_{3,2}, \bar{d}_{2,3}$
$d_{3,1}$	$\{D, E, F\}$	$A_{3,8}$	$d_{4,1}$	$D^2 - E^2 - F^2$	$\bar{d}_{2,4}, \bar{d}_{1,2}$
$\bar{d}_{2,2} = a_{3,5}$	$C \oplus \{F; D + E\}$	$A_1 \oplus A_2$	self	$C$	$\bar{d}_{2,1}, \bar{d}_{2,2}, \bar{d}_{2,4}, \bar{d}_{2,5}$
$\bar{d}_{2,1} = a_{2,7}$	$C, F$	$2A_1$	self	$C, F$	$d_{1,1}, \bar{d}_{1,4}, \bar{d}_{1,6}$
$\bar{d}_{2,2} = a_{2,3}$	$C, D + E$	$2A_1$	$\bar{d}_{3,2}$	$C, D + E$	$d_{1,1}, \bar{d}_{1,5}, \bar{d}_{1,7}$
$\bar{d}_{2,3} = c_{2,4}$	$C, D$	$2A_1$	self	$C, D$	$d_{1,1}, \bar{d}_{1,2}, \bar{d}_{1,3}, \bar{d}_{1,8}$
$\bar{d}_{2,4} = a_{2,13}$	$F; D + E$	$A_2$	$\bar{d}_{3,2}$	none	$\bar{d}_{1,4}, \bar{d}_{1,5}$
$\bar{d}_{2,5}^d = a_{2,17}^d$ $d > 0 [d \neq 0]$	$F + dC; D + E$	$A_2$	$\bar{d}_{3,2}$	none	$\bar{d}_{1,5}, \bar{d}_{1,6}$
$\bar{d}_{1,1}$	$C$	$A_1$	$d_{4,1}$	$C$	
$\bar{d}_{1,2} = c_{1,1}$	$D$	$A_1$	$c_{4,1}$	$D$	
$\bar{d}_{1,3}^e = c_{1,2}^e$ $e > 0, \neq 1 [e \neq 0, \pm 1]$	$D + eC$	$A_1$	$\bar{d}_{2,3}$	$D + eC$	
$\bar{d}_{1,4} = a_{1,1}$	$F$	$A_1$	$a_{4,4}$	$F$	
$\bar{d}_{1,5} = a_{1,3}$	$D + E$	$A_1$	$a_{5,7}$	$D + E$	
$\bar{d}_{1,6}^e = a_{1,7}^e, e > 0$	$F + eC$	$A_1$	$a_{2,7}$	$F + eC$	
$\bar{d}_{1,7}^e = a_{1,6}^e$ $\epsilon = 1 [\epsilon = \pm 1]$	$C + \epsilon(D + E)$	$A_1$	$a_{2,3}$	$C + \epsilon(D + E)$	
$\bar{d}_{1,8}^e \sim e_{1,1}$ $\epsilon = 1 [\epsilon = \pm 1]$	$D + \epsilon C$	$A_1$	$\sim e_{4,1}$	$D + \epsilon C$	

TABLE VIII. Subalgebras of  $LO(2,2)$ .

Name and range of parameters	Generators	Isomorphism class	Normalizer	Invariants	Maximal subalgebras
$e_{6,1}$	$\{A_1, A_2, A_3\} \oplus \{B_1, B_2, B_3\}$	$A_{3,8} \oplus A_{3,8}$	self	$A_1^2 + A_2^2 - A_3^2, B_1^2 + B_2^2 - B_3^2$	$\bar{e}_{5,1}, \bar{e}_{5,1}, e_{4,1}, \bar{e}_{4,1}, \bar{e}_{3,8}, \bar{e}_{3,9}$
$\bar{e}_{5,1} = b_{5,1}$	$\{A_2; A_1 - A_3\} \oplus \{B_1, B_2, B_3\}$	$A_2 \oplus A_{3,8}$	self	$B_1^2 + B_2^2 - B_3^2$	$\bar{e}_{4,2}, \bar{e}_{4,3}, \bar{e}_{4,4}, \bar{e}_{3,4}$
$\bar{e}_{5,1}$	$\{A_1, A_2, A_3\} \oplus \{B_2; B_1 - B_3\}$	$A_{3,8} \oplus A_2$	self	$A_1^2 + A_2^2 - A_3^2$	$\bar{e}_{4,2}, \bar{e}_{4,3}, \bar{e}_{4,4}, \bar{e}_{3,4}$
$e_{4,1}$	$A_3 \oplus \{B_1, B_2, B_3\}$	$A_1 \oplus A_{3,8}$	self	$A_3, B_1^2 + B_2^2 - B_3^2$	$e_{3,1}, \bar{e}_{3,4}, \bar{e}_{2,6}$
$\bar{e}_{4,1}$	$\{A_1, A_2, A_3\} \oplus B_3$	$A_{3,8} \oplus A_1$	self	$A_1^2 + A_2^2 - A_3^2, B_3$	$\bar{e}_{3,1}, \bar{e}_{3,4}, \bar{e}_{2,6}$

TABLE VIII. (continued)

Name and range of parameters	Generators	Isomorphism class	Normalizer	Invariants	Maximal subalgebras
$\bar{e}_{4,2} = a_{4,6}$	$\{A_2; A_1 - A_3\}$ $\oplus \{B_2; B_1 - B_3\}$	$A_2 \oplus A_2$	self	none	$\bar{e}_{3,2}, \bar{e}_{3,2}, \bar{e}_{3,3}, \bar{e}_{3,3},$ $\bar{e}_{3,5}, \bar{e}_{3,6}, \bar{e}_{3,7}$
$\bar{e}_{4,3} = b_{4,1}$	$(A_1 - A_3) \oplus$ $\{B_1, B_2, B_3\}$	$A_1 \oplus A_{3,8}$	$\bar{e}_{3,1}$	$A_1 - A_3, B_1^2 + B_2^2 - B_3^2$	$e_{3,1}, \bar{e}_{3,3}, \bar{e}_{2,4}$
$\bar{e}_{4,3}$	$\{A_1, A_2, A_3\}$ $\oplus (B_1 - B_3)$	$A_{3,8} \oplus A_1$	$\sim \bar{e}_{5,1}$	$A_1^2 + A_2^2 - A_3^2, B_1 - B_3$	$\bar{e}_{3,1}, \bar{e}_{3,3}, \bar{e}_{2,4}$
$\bar{e}_{4,4} = b_{4,2}$	$A_2 \oplus \{B_1, B_2, B_3\}$	$A_1 \oplus A_{3,8}$	self	$A_2, B_1^2 + B_2^2 - B_3^2$	$e_{3,1}, \bar{e}_{3,2}, \bar{e}_{2,5}$
$\bar{e}_{4,4}$	$\{A_1, A_2, A_3\} \oplus B_2$	$A_{3,8} \oplus A_1$	self	$A_1^2 + A_2^2 - A_3^2, B_2$	$\bar{e}_{3,1}, \bar{e}_{3,2}, \bar{e}_{2,5}$
$e_{3,1} = \bar{b}_{3,3}$	$\{B_1, B_2, B_3\}$	$A_{3,8}$	$e_{6,1}$	$B_1^2 + B_2^2 - B_3^2$	$\bar{e}_{2,7}, \bar{e}_{1,1}$
$\bar{e}_{3,1}$	$\{A_1, A_2, A_3\}$	$A_{3,8}$	$e_{6,1}$	$A_1^2 + A_2^2 - A_3^2$	$\bar{e}_{2,7}, e_{1,1}$
$\bar{e}_{3,2} = a_{3,4}$	$B_2 \oplus \{A_2; A_1 - A_3\}$	$A_1 \oplus A_2$	self	$B_2$	$\bar{e}_{2,2}, \bar{e}_{2,3}, \bar{e}_{2,7}, \bar{e}_{2,10},$ $\bar{e}_{2,11}$
$\bar{e}_{3,2}$	$A_2 \oplus \{B_2; B_1 - B_3\}$	$A_1 \oplus A_2$	self	$A_2$	$\bar{e}_{2,2}, \bar{e}_{2,3}, \bar{e}_{2,7}, \bar{e}_{2,10},$ $\bar{e}_{2,11}$
$\bar{e}_{3,3} = a_{3,6}$	$(A_1 - A_3) \oplus$ $\{B_2; B_1 - B_3\}$	$A_1 \oplus A_2$	$\bar{e}_{4,2}$	$A_1 - A_3$	$\bar{e}_{2,1}, \bar{e}_{2,3}, \bar{e}_{2,7}, \bar{e}_{2,13}$
$\bar{e}_{3,3}$	$(B_1 - B_3) \oplus$ $\{A_2; A_1 - A_3\}$	$A_1 \oplus A_2$	$\bar{e}_{4,2}$	$B_1 - B_3$	$\bar{e}_{2,1}, \bar{e}_{2,3}, \bar{e}_{2,7}, \bar{e}_{2,13}$
$\bar{e}_{3,4} = b_{3,1}$	$B_3 \oplus \{A_2; A_1 - A_3\}$	$A_1 \oplus A_2$	self	$B_3$	$\bar{e}_{2,4}, \bar{e}_{2,5}, \bar{e}_{2,7}, \bar{e}_{2,12}$
$\bar{e}_{3,4}$	$A_3 \oplus \{B_2; B_1 - B_3\}$	$A_1 \oplus A_2$	self	$A_3$	$\bar{e}_{2,4}, \bar{e}_{2,5}, \bar{e}_{2,7}, \bar{e}_{2,12}$
$\bar{e}_{3,5} = a_{3,12}$	$A_2 + B_2; A_1 - A_3,$ $B_1 - B_3$	$A_{3,3}$	$\bar{e}_{4,2}$	$(A_1 - A_3)/(B_1 - B_3)$	$\bar{e}_{2,1}, \bar{e}_{2,8}, \bar{e}_{2,9}, \bar{e}_{2,10},$ $\bar{e}_{2,10}$
$\bar{e}_{3,6} = a_{3,13}$	$A_2 - B_2; A_1 - A_3,$ $B_1 - B_3$	$A_{3,4}$	$a_{5,6}$	$(A_1 - A_3)(B_1 - B_3)$	$\bar{e}_{2,1}, \bar{e}_{2,10}, \bar{e}_{2,10}$
$\bar{e}_{3,7} = d_{3,20}^{(1-c)/(1+c)}$ $0 <  c  < 1 [c \neq 0, \pm 1]$	$A_2 + cB_2;$ $A_1 - A_3, B_1 - B_3$	$A_{3,5}^h$ $h = c,  c  < 1$ $h = 1/c,  c  > 1$	$\bar{e}_{4,2}$	$(A_1 - A_3)^c (B_1 - B_3)$	$\bar{e}_{2,1}, \bar{e}_{2,11}, \bar{e}_{2,11}$
$\bar{e}_{3,8} = d_{3,1}$	$\{A_1 - B_1,$ $A_2 + B_2, A_3 - B_3\}$	$A_{3,8}$	$d_{4,1}$	$(A_1 - B_1)^2 + (A_2 + B_2)^2$ $- (A_3 - B_3)^2$	$\bar{e}_{2,9}, \bar{e}_{1,13}$
$\bar{e}_{3,9} \sim a_{3,24}$	$\{A_1 + B_1, A_2 + B_2,$ $A_3 + B_3\}$	$A_{3,8}$	$\sim a_{4,4}$	$(A_1 + B_1)^2 + (A_2 + B_2)^2$ $- (A_3 + B_3)^2$	$\bar{e}_{2,8}, \bar{e}_{1,14}$
$\bar{e}_{2,1} = a_{2,5}$	$A_1 - A_3, B_1 - B_3$	$2A_1$	$a_{5,6}$	$A_1 - A_3, B_1 - B_3$	$\bar{e}_{1,3}, \bar{e}_{1,4}, \bar{e}_{1,10}, \bar{e}_{1,10}$
$\bar{e}_{2,2} = a_{2,6}$	$A_2, B_2$	$2A_1$	self	$A_2, B_2$	$\bar{e}_{1,2}, \bar{e}_{1,5}, \bar{e}_{1,11}, \bar{e}_{1,11}$
$\bar{e}_{2,3} = a_{2,8}$	$B_2, A_1 - A_3$	$2A_1$	$\bar{e}_{3,2}$	$B_2, A_1 - A_3$	$\bar{e}_{1,6}, \bar{e}_{1,10}, \bar{e}_{1,11}$
$\bar{e}_{2,3}$	$A_2, B_1 - B_3$	$2A_1$	$\bar{e}_{3,2}$	$A_2, B_1 - B_3$	$\bar{e}_{1,6}, \bar{e}_{1,10}, \bar{e}_{1,11}$
$\bar{e}_{2,4} = b_{2,1}$	$A_1 - A_3, B_3$	$2A_1$	$\bar{e}_{3,4}$	$A_1 - A_3, B_3$	$\bar{e}_{1,1}, \bar{e}_{1,7}, \bar{e}_{1,8}, \bar{e}_{1,10}$
$\bar{e}_{2,4}$	$A_3, B_1 - B_3$	$2A_1$	$\bar{e}_{3,4}$	$A_3, B_1 - B_3$	$e_{1,1}, \bar{e}_{1,7}, \bar{e}_{1,8}, \bar{e}_{1,10}$
$\bar{e}_{2,5} = b_{2,2}$	$A_2, B_3$	$2A_1$	self	$A_2, B_3$	$\bar{e}_{1,1}, \bar{e}_{1,9}, \bar{e}_{1,11}$
$\bar{e}_{2,5}$	$A_3, B_2$	$2A_1$	self	$A_3, B_2$	$e_{1,1}, \bar{e}_{1,9}, \bar{e}_{1,11}$
$\bar{e}_{2,6} \sim c_{2,1}$	$A_3, B_3$	$2A_1$	self	$A_3, B_3$	$e_{1,1}, \bar{e}_{1,1}, \bar{e}_{1,12}, \bar{e}_{1,13},$ $\bar{e}_{1,14}$
$\bar{e}_{2,7} = b_{2,4}$	$A_2; A_1 - A_3$	$A_2$	$\bar{e}_{5,1}$	none	$\bar{e}_{1,10}, \bar{e}_{1,11}$
$\bar{e}_{2,7}$	$B_2; B_1 - B_3$	$A_2$	$\bar{e}_{5,1}$	none	$\bar{e}_{1,10}, \bar{e}_{1,11}$
$\bar{e}_{2,8} \sim \bar{a}_{2,10}$	$A_2 + B_2;$ $-A_1 + A_3 - B_1 + B_3$	$A_2$	$\sim \bar{a}_{3,3}$	none	$\bar{e}_{1,2}, \bar{e}_{1,3}$

TABLE VIII. (continued)

Name and range of parameters	Generators	Isomorphism class	Normalizer	Invariants	Maximal subalgebras
$\bar{e}_{2,9} = a_{2,13}$	$A_2 + B_2; -A_1 + A_3 + B_1 - B_3$	$A_2$	$a_{3,5}$	none	$\bar{e}_{1,2}, \bar{e}_{1,4}$
$\bar{e}_{2,10} = \tilde{a}_{2,11}$	$A_2 + B_2; -A_1 + A_3$	$A_2$	$\tilde{a}_{4,5}$	none	$\bar{e}_{1,2}, \bar{e}_{1,10}$
$\bar{e}_{2,10} \cong \bar{e}_{2,10}$	$A_2 + B_2; -B_1 + B_3$	$A_2$	$\sim \tilde{a}_{4,5}$	none	$\bar{e}_{1,2}, \bar{e}_{1,10}$
$\bar{e}_{2,11}^d = a_{2,13}^{d/(1+d)}$ $d > 0, \neq 1$	$A_2 + dB_2; -A_1 + A_3$	$A_2$	$\bar{e}_{3,2}$	none	$\bar{e}_{1,5}, \bar{e}_{1,10}$
$\bar{e}_{2,11}^d \cong \bar{e}_{2,11}^d$ $d > 0, \neq 1$	$A_2 + dB_2; -B_1 + B_3$	$A_2$	$\cong \bar{e}_{3,2}$	none	$\bar{e}_{1,5}, \bar{e}_{1,10}$
$\bar{e}_{2,12}^d = b_{2,3}^d$ $d > 0 [d \neq 0]$	$A_2 - dB_3; -A_1 + A_3$	$A_2$	$e_{3,4}$	none	$\bar{e}_{1,9}, \bar{e}_{1,10}$
$\bar{e}_{2,12}^d \cong \bar{e}_{2,12}^d$ $d > 0 [d \neq 0]$	$B_2 - dA_3; -B_1 + B_3$	$A_2$	$\cong e_{3,4}$	none	$\bar{e}_{1,9}, \bar{e}_{1,10}$
$\bar{e}_{2,13}^\epsilon = a_{2,13}^\epsilon$ $\epsilon = 1 [\epsilon = \pm 1]$	$B_2 + \epsilon(A_3 - A_1); B_1 - B_3$	$A_2$	$\bar{e}_{3,3}$	none	$\bar{e}_{1,6}, \bar{e}_{1,10}$
$\bar{e}_{2,13}^\epsilon \cong \bar{e}_{2,13}^\epsilon$ $\epsilon = 1 [\epsilon = \pm 1]$	$A_2 + \epsilon(B_3 - B_1); A_1 - A_3$	$A_2$	$\cong \bar{e}_{3,3}$	none	$\bar{e}_{1,6}, \bar{e}_{1,10}$
$e_{1,1}$	$A_3$	$A_1$	$e_{4,1}$	$A_3$	
$\tilde{e}_{1,1}$	$B_3$	$A_1$	$\tilde{e}_{4,1}$	$B_3$	
$\bar{e}_{1,2} = a_{1,1}$	$A_2 + B_2$	$A_1$	$a_{4,4}$	$A_2 + B_2$	
$\bar{e}_{1,3} \sim a_{1,2}$	$-A_1 + A_3 - B_1 + B_3$	$A_1$	$\sim a_{5,6}$	$-A_1 + A_3 - B_1 + B_3$	
$\bar{e}_{1,4} = a_{1,3}$	$-A_1 + A_3 + B_1 - B_3$	$A_1$	$a_{5,7}$	$-A_1 + A_3 + B_1 - B_3$	
$\bar{e}_{1,5}^e = a_{1,8}^{e/(1+e)}$ , $0 < e < 1$ $[e > 0, \neq 1]$	$A_2 + eB_2$	$A_1$	$\bar{e}_{2,2}$	$A_2 + eB_2$	
$\bar{e}_{1,6} = a_{1,9}$	$B_2 - A_1 + A_3$	$A_1$	$\bar{e}_{2,3}$	$B_2 - A_1 + A_3$	
$\bar{e}_{1,6} \cong \bar{e}_{1,6}$	$A_2 - B_1 + B_3$	$A_1$	$\cong \bar{e}_{2,3}$	$A_2 - B_1 + B_3$	
$\bar{e}_{1,7} = b_{1,1}$	$-A_1 + A_3 + B_3$	$A_1$	$\bar{e}_{2,4}$	$-A_1 + A_3 + B_3$	
$\bar{e}_{1,7} \cong \bar{e}_{1,7}$	$-B_1 + B_3 + A_3$	$A_1$	$\cong \bar{e}_{2,4}$	$-B_1 + B_3 + A_3$	
$\bar{e}_{1,8} = b_{1,2}$	$-A_1 + A_3 - B_3$	$A_1$	$\bar{e}_{2,4}$	$-A_1 + A_3 - B_3$	
$\bar{e}_{1,8} \cong \bar{e}_{1,8}$	$-B_1 + B_3 - A_3$	$A_1$	$\cong \bar{e}_{2,4}$	$-B_1 + B_3 - A_3$	
$\bar{e}_{1,9}^e = b_{1,3}$ , $e > 0$	$A_2 - eB_3$	$A_1$	$\bar{e}_{2,5}$	$A_2 - eB_3$	
$\bar{e}_{1,9}^e \cong \bar{e}_{1,9}^e$ , $e > 0$	$B_2 - eA_3$	$A_1$	$\cong \bar{e}_{2,5}$	$B_2 - eA_3$	
$\bar{e}_{1,10} = b_{1,4}$	$-A_1 + A_3$	$A_1$	$b_{7,1}$	$-A_1 + A_3$	
$\bar{e}_{1,10} \cong \bar{e}_{1,10}$	$-B_1 + B_3$	$A_1$	$\sim b_{7,1}$	$-B_1 + B_3$	
$\bar{e}_{1,11} = b_{1,5}$	$A_2$	$A_1$	$\bar{e}_{4,4}$	$A_2$	
$\bar{e}_{1,11} \cong \bar{e}_{1,11}$	$B_2$	$A_1$	$\cong \bar{e}_{4,4}$	$B_2$	
$\bar{e}_{1,12}^e \sim c_{1,2}^{e/(1+e)}$ , $0 <  e  < 1$ $[e \neq 0, \pm 1]$	$A_3 + eB_3$	$A_1$	$\bar{e}_{2,6}$	$A_3 + eB_3$	
$\bar{e}_{1,13} = c_{1,1}$	$A_3 - B_3$	$A_1$	$c_{4,1}$	$A_3 - B_3$	
$\bar{e}_{1,14} \sim d_{1,1}$	$A_3 + B_3$	$A_1$	$d_{4,1}$	$A_3 + B_3$	



TABLE IX. Subalgebras of  $LO(3,1)$ .

Name and range of parameters	Generators	Isomorphism class	Normalizer	Invariants	Maximal subalgebras
$f_{6,1}$	$;L_1, L_2, L_3, K_1, K_2, K_3$	$LO(3,1)$	self	$L^2 - K^2, (L, K)$	$\bar{f}_{4,1}, \bar{f}_{3,4}, \bar{f}_{3,5}$
$\bar{f}_{4,1} = a_{4,17}$	$K_1, L_1; L_2 - K_3, L_3 + K_2$	$A_{4,12}$	self	none	$\bar{f}_{3,1}, \bar{f}_{3,2}, \bar{f}_{3,3}, \bar{f}_{2,2}$
$\bar{f}_{3,1} = a_{3,11}$	$K_1; L_2 - K_3, L_3 + K_2$	$A_{3,3}$	$f_{4,1}$	$(L_2 - K_3)/(L_3 + K_2)$	$\bar{f}_{2,1}, \bar{f}_{2,3}$
$\bar{f}_{3,2} = a_{3,21}$	$L_1; L_2 - K_3, L_3 + K_2$	$A_{3,6}$	$a_{5,7}$	$(L_2 - K_3)^2 + (L_3 + K_2)^2$	$\bar{f}_{2,1}, \bar{f}_{1,1}$
$\bar{f}_{3,3}^c = a_{3,23}$ $c > 0 [c \neq 0]$	$K_1 - cL_1; L_2 - K_3, L_3 + K_2$	$A_{3,7}^{1/2}$	$\bar{f}_{4,1}$	$[(L_2 - K_3)^2 - (L_3 + K_2)^2]^c \times \left( \frac{L_2 - K_3 - i(L_3 + K_2)}{L_2 - K_3 + i(L_3 + K_2)} \right)^i$	$\bar{f}_{2,1}, \bar{f}_{1,4}^c$
$\bar{f}_{3,4} \sim a_{3,24}$	$;K_2, K_3, L_1$	$A_{3,8}$	$a_{4,4}$	$L_1^2 - K_2^2 - K_3^2$	$\bar{f}_{2,3}, \bar{f}_{1,1}$
$\bar{f}_{3,5} = c_{3,1}$	$;L_1, L_2, L_3$	$A_{3,9}$	$c_{4,1}$	$L_1^2 + L_2^2 + L_3^2$	$\bar{f}_{1,1}$
$\bar{f}_{2,1} = a_{2,4}$	$L_2 - K_3, L_3 + K_2$	$2A_1$	$a_{5,7}$	$L_2 - K_3, L_3 + K_2$	$\bar{f}_{1,3}$
$\bar{f}_{2,2} = a_{2,7}$	$L_1, K_1$	$2A_1$	self	$L_1, K_1$	$\bar{f}_{1,1}, \bar{f}_{1,2}, \bar{f}_{1,4}^e$
$\bar{f}_{2,3} = \tilde{a}_{2,10}$	$K_1; L_2 - K_3$	$A_2$	$\tilde{a}_{3,3}$	none	$\bar{f}_{1,2}, \bar{f}_{1,3}$
$\bar{f}_{1,1} = d_{1,1}$	$L_1$	$A_1$	$d_{4,1}$	$L_1$	
$\bar{f}_{1,2} = a_{1,1}$	$K_1$	$A_1$	$a_{4,4}$	$K_1$	
$\bar{f}_{1,3} = a_{1,2}$	$L_2 - K_3$	$A_1$	$a_{5,6}$	$L_2 - K_3$	
$\bar{f}_{1,4}^e = a_{1,7}^e, e > 0 [e \neq 0]$	$K_1 - eL_1$	$A_1$	$\bar{f}_{2,2}$	$K_1 - eL_1$	

TABLE X. Subalgebras of the irreducibly embedded  $LO(2,1)$ .

Name	Generators	Isomorphism class	Normalizer	Invariants
$g_{3,1}$	$;K_1, K_2, L_3$	$A_{3,8}$	self	$K_1^2 + K_2^2 - L_3^2$
$\bar{g}_{2,1} \sim a_{2,20}^1$	$K_2; K_1 - L_3$	$A_2$	self	none
$\bar{g}_{1,1} \sim c_{1,2}^2$	$L_3$	$A_1$	$\sim c_{2,1}$	$L_3$
$\bar{g}_{1,2} \sim a_{1,5}^1$	$K_1 - L_3$	$A_1$	$\sim a_{3,18}^1$	$K_1 - L_3$
$\bar{g}_{1,3} \sim a_{1,8}^{1/2}$	$K_2$	$A_1$	$\sim a_{2,6}$	$K_2$

?!

Not ***even*** UG students  
should have to work so hard.

# HOMOGENEOUS LORENTZIAN MANIFOLDS

(the smart way)

We observe that  $SO(N) \times SO(3,2)$  is **semisimple** for  $N > 2$ .

## Theorem

Let  $M$  be a lorentzian manifold with the transitive and isometric action of a semisimple Lie group  $G$ .

Then **either**

- (1) the action of  $G$  is proper; **or**
- (2)  $M$  is the product of (anti) de Sitter space with a riemannian homogeneous manifold.

Kowalsky (1996)

Deffaf+Melnick+Zeghib (2008)



✓ We already classified Freund-Rubin AdS backgrounds. (Still need to check AdS backgrounds which are **not** Freund-Rubin.)

- ✓ We already classified Freund-Rubin AdS backgrounds. (Still need to check AdS backgrounds which are **not** Freund-Rubin.)
- ✓ There are no dS backgrounds.

- ✓ We already classified Freund-Rubin AdS backgrounds. (Still need to check AdS backgrounds which are **not** Freund-Rubin.)
- ✓ There are no dS backgrounds.
- ✓ If the action of  $G$  is proper, then  $H$  is compact.



# Revised problem

**Classify** compact Lie subalgebras

$$\mathfrak{h} \subset \mathfrak{so}(N) \oplus \mathfrak{so}(3, 2) \quad \dim \mathfrak{h} = \binom{N}{2} - 1$$

for  $N \geq 5$ , leaving invariant a lorentzian inner product on  $\mathfrak{g}/\mathfrak{h}$ .

In particular,  $\mathfrak{h} \subset \mathfrak{so}(N) \oplus \mathfrak{so}(3) \oplus \mathfrak{so}(2)$

Because  $\mathfrak{h}$  is compact, it is reductive, hence

$$\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}$$

where  $[\mathfrak{h}, \mathfrak{m}] \subset \mathfrak{m}$ . Also  $\mathfrak{h}$  leaves invariant a positive-definite inner product  $\beta$  on  $\mathfrak{m}$ , and every  $\mathfrak{h}$ -invariant lorentzian inner product on  $\mathfrak{m}$  is of the form

$$\beta - \alpha \otimes \alpha$$

for some  $\mathfrak{h}$ -invariant  $\alpha$  in  $\mathfrak{m}$ .

*See, e.g., Alekseevsky (2011)*

In particular, unless there are nonzero  $\mathfrak{h}$ -invariants in  $\mathfrak{m}$ , there is no  $\mathfrak{h}$ -invariant lorentzian inner product either.

# **What we did last summer..**

# What we did last summer...

- ✓ Classified compact Lie subalgebras of  $\mathfrak{so}(3,2)$

# What we did last summer...

- ✓ Classified compact Lie subalgebras of  $\mathfrak{so}(3,2)$
- ✓ Classified compact Lie subalgebras of  $\mathfrak{so}(N)$  for  $N=4,5,6,7$ . (No need to do  $N=8$ .)

# What we did last summer...

- ✓ Classified compact Lie subalgebras of  $\mathfrak{so}(3,2)$
- ✓ Classified compact Lie subalgebras of  $\mathfrak{so}(N)$  for  $N=4,5,6,7$ . (No need to do  $N=8$ .)
- ✓ Classified fibred products of the right dimension yielding all possible  $\mathfrak{h}$ .

# What we did last summer...

- ✓ Classified compact Lie subalgebras of  $\mathfrak{so}(3,2)$
- ✓ Classified compact Lie subalgebras of  $\mathfrak{so}(N)$  for  $N=4,5,6,7$ . (No need to do  $N=8$ .)
- ✓ Classified fibred products of the right dimension yielding all possible  $\mathfrak{h}$ .
- ✓ Determined  $\mathfrak{h}$ -invariant lorentzian inner products and  $\mathfrak{h}$ -invariant 4-forms.

$$N=7$$

There are no compact subalgebras of the right dimension, assuming that  $\mathfrak{so}(3,2)$  acts effectively.

Otherwise  $\mathfrak{so}(3,2)$  acts trivially, in which case we can take  $SO(7)/SO(5)$ , which does have an invariant lorentzian metric and even invariant fluxes, but they do **not** satisfy the supergravity field equations.





There are no compact subalgebras of the right dimension, assuming that  $\mathfrak{so}(3,2)$  acts effectively.

Otherwise  $\mathfrak{so}(3,2)$  acts trivially, in which case we can take  $SO(7)/SO(5)$ , which does have an invariant lorentzian metric and even invariant fluxes, but they do **not** satisfy the supergravity field equations.

$$N=6$$

If  $\mathfrak{so}(3,2)$  acts effectively, then

$$\mathfrak{h} = \mathfrak{so}(5) \oplus \mathfrak{so}(3) \oplus \mathfrak{so}(2),$$

but there is no lorentzian invariant metric on such a homogeneous space.

If  $\mathfrak{so}(3,2)$  acts trivially,  $\mathfrak{h} \cong \mathfrak{so}(3) \oplus \mathfrak{so}(2)$ . There are 5 inequivalent  $\mathfrak{so}(3) \oplus \mathfrak{so}(2)$  subalgebras of  $\mathfrak{so}(6)$ , 3 of which have invariant lorentzian metrics and invariant fluxes. (Still must check the field equations.)



If  $\mathfrak{so}(3,2)$  acts effectively, then

$$\mathfrak{h} = \mathfrak{so}(5) \oplus \mathfrak{so}(3) \oplus \mathfrak{so}(2),$$

but there is no lorentzian invariant metric on such a homogeneous space.

If  $\mathfrak{so}(3,2)$  acts trivially,  $\mathfrak{h} \cong \mathfrak{so}(3) \oplus \mathfrak{so}(2)$ . There are 5 inequivalent  $\mathfrak{so}(3) \oplus \mathfrak{so}(2)$  subalgebras of  $\mathfrak{so}(6)$ , 3 of which have invariant lorentzian metrics and invariant fluxes. (Still must check the field equations.)

$$N=5$$

Here  $\mathfrak{so}(3,2)$  has to act effectively, by dimension.

There is precisely one possible subalgebra of the right dimension:

$$\mathfrak{h} = \mathfrak{so}(4) \oplus \mathfrak{so}(3)$$

and there are invariant lorentzian inner products.

The resulting geometry is

$$S^4 \times X^7$$

$$X^7 \cong \mathrm{SO}(3, 2) / \mathrm{SO}(3)$$

HOMOGENEOUS

$N=5$  (?)

BACKGROUNDS

# **d=11 supergravity backgrounds**

$$(M, g, F)$$

$(M^{11}, g)$  lorentzian spin manifold

$$F \in \Omega^4(M) \quad dF = 0$$

$$d \star F = \frac{1}{2} F \wedge F$$

$$\text{Ric}(X, Y) = \frac{1}{2} \langle \iota_X F, \iota_Y F \rangle - \frac{1}{6} |F|^2 g(X, Y)$$

**Not unlike Einstein-Maxwell theory**

# Homogeneous backgrounds

$$M = G/H$$

$\mathfrak{g} \leftrightarrow$   $H$ -invariant inner product on  $\mathfrak{g}/\mathfrak{h}$

$F \leftrightarrow$   $H$ -invariant 4-form on  $\mathfrak{g}/\mathfrak{h}$

Field equations become algebraic equations in these data.

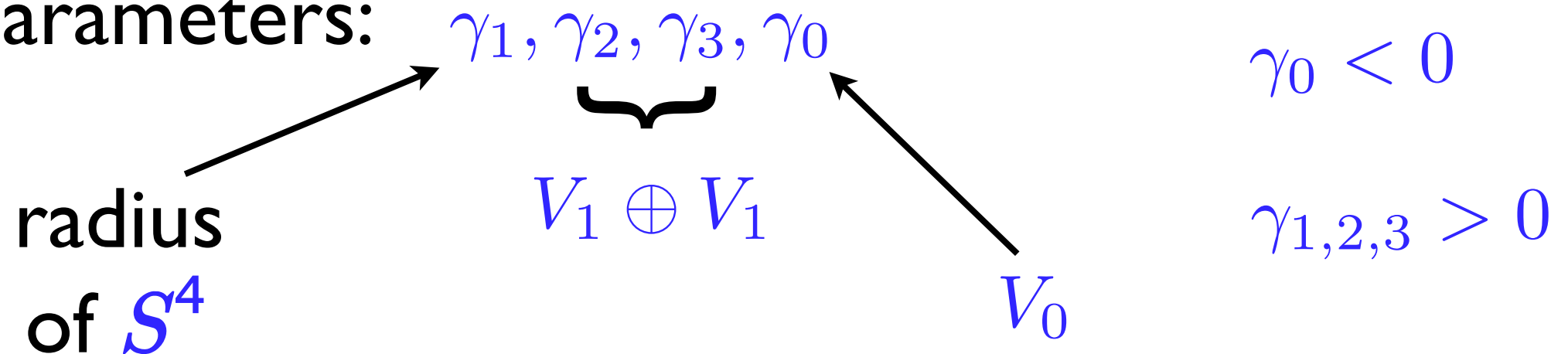
The backgrounds in question are all of the form

$$S^4 \times X^7 \quad X^7 \cong \text{SO}(3, 2)/\text{SO}(3)$$

$$\mathfrak{so}(3, 2) = \mathfrak{so}(3) \oplus \mathfrak{m} \quad \mathfrak{m} = V_1 \oplus V_1 \oplus V_0$$

$V_s$  = irreducible spin  $s$  module of  $\mathfrak{so}(3)$

The most general lorentzian metric has four parameters:



Used normaliser of  $\mathfrak{so}(3)$  in  $\mathfrak{so}(3,2)$  to diagonalise inner product on  $V_1 \oplus V_1$



There is a 6-dimensional space of invariant 4-forms:

$$(\Lambda^4 \mathfrak{m})^{\text{SO}(3)} = \mathbb{R}\omega^2 \oplus S^3\mathbb{R}^2 \otimes (\Lambda^3 V_1)^{\text{SO}(3)} \otimes V_0$$

and the volume form on  $S^4$

$\omega = \text{SO}(3)$ -invariant symplectic form on  $V_1 \oplus V_1$

Finally we use the homothety symmetry

$$g \mapsto t^2 g \quad F \mapsto t^3 F$$

to set  $\gamma_0 = -1$

In summary, we have a number of (almost polynomial) equations on 9 real parameters:

$$\gamma_1, \gamma_2, \gamma_3, \varphi_1, \dots, \varphi_6 \quad \gamma_{1,2,3} > 0$$

The Maxwell equations:

$$\varphi_4 = \frac{\gamma_2^{1/2} \varphi_1 \varphi_3}{3\gamma_1^2 \gamma_3^{1/2}}$$

$$\varphi_5 = \frac{\gamma_3^{1/2} \varphi_1 \varphi_2}{3\gamma_1^2 \gamma_2^{1/2}}$$

$$0 = \left( \frac{\varphi_1}{\gamma_1^2} + \frac{2}{\gamma_2^{1/2} \gamma_3^{1/2}} \right) \varphi_6 .$$

$$\frac{\gamma_2 \varphi_3}{\gamma_3} = \left( \frac{2\gamma_3}{\gamma_2} + \frac{\gamma_3^{1/2} \varphi_1}{\gamma_1^2 \gamma_2^{1/2}} \right) \varphi_4$$

$$\frac{\gamma_3 \varphi_2}{\gamma_2} = \left( \frac{2\gamma_2}{\gamma_3} + \frac{\gamma_2^{1/2} \varphi_1}{\gamma_1^2 \gamma_3^{1/2}} \right) \varphi_5$$

In summary, we have a number of (almost polynomial) equations on 9 real parameters:

$$\gamma_1, \gamma_2, \gamma_3, \varphi_1, \dots, \varphi_6 \quad \gamma_{1,2,3} > 0$$

The Maxwell equations:

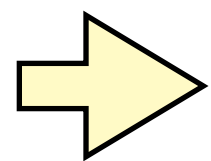
$$\varphi_4 = \frac{\gamma_2^{1/2} \varphi_1 \varphi_3}{3\gamma_1^2 \gamma_3^{1/2}}$$

$$\frac{\gamma_2 \varphi_3}{\gamma_3} = \left( \frac{2\gamma_3}{\gamma_2} + \frac{\gamma_3^{1/2} \varphi_1}{\gamma_1^2 \gamma_2^{1/2}} \right) \varphi_4$$

$$\varphi_5 = \frac{\gamma_3^{1/2} \varphi_1 \varphi_2}{3\gamma_1^2 \gamma_2^{1/2}}$$

$$\frac{\gamma_3 \varphi_2}{\gamma_2} = \left( \frac{2\gamma_2}{\gamma_3} + \frac{\gamma_2^{1/2} \varphi_1}{\gamma_1^2 \gamma_3^{1/2}} \right) \varphi_5$$

$$0 = \left( \frac{\varphi_1}{\gamma_1^2} + \frac{2}{\gamma_2^{1/2} \gamma_3^{1/2}} \right) \varphi_6 .$$



$$\varphi_3 = \frac{1}{3} \left( \frac{2\gamma_3}{\gamma_2} + \frac{\gamma_3^{1/2} \varphi_1}{\gamma_1^2 \gamma_2^{1/2}} \right) \frac{\gamma_3^{1/2} \varphi_1}{\gamma_1^2 \gamma_2^{1/2}} \varphi_3 \quad \text{and} \quad \varphi_2 = \frac{1}{3} \left( \frac{2\gamma_2}{\gamma_3} + \frac{\gamma_2^{1/2} \varphi_1}{\gamma_1^2 \gamma_3^{1/2}} \right) \frac{\gamma_2^{1/2} \varphi_1}{\gamma_1^2 \gamma_3^{1/2}} \varphi_2$$

# The Einstein equations:

$$0 = \varphi_1 \varphi_2 \varphi_3 (3\gamma_1^2(\gamma_2 + \gamma_3) + 2\sqrt{\gamma_2 \gamma_3} \varphi_1)$$

$$0 = 3\gamma_1^4 \varphi_6^2 + \gamma_2^2 \gamma_3^2 (12\gamma_1^3 - \varphi_1^2) + 3\gamma_1^4 \gamma_2 \gamma_3 ((\gamma_2 - \gamma_3)^2 - 1)$$

$$0 = (3\gamma_2 \gamma_1^4 + \gamma_3 \varphi_1^2) \gamma_2^2 \varphi_3^2 + (3\gamma_3 \gamma_1^4 + \gamma_2 \varphi_1^2) \gamma_3^2 \varphi_2^2 - 9\gamma_1^4 \gamma_2 \gamma_3 \varphi_6^2 + 6\gamma_2^3 \gamma_3^3 \varphi_1^2 - 54\gamma_1^3 \gamma_2^3 \gamma_3^3$$

$$0 = 6\gamma_1^4 \gamma_2^3 \varphi_3^2 + (\gamma_2 \varphi_1^2 - 3\gamma_1^4 \gamma_3) \gamma_3^2 \varphi_2^2 - 9\gamma_1^4 \gamma_2 \gamma_3 \varphi_6^2 + 3\gamma_2^3 \gamma_3^3 \varphi_1^2 + 9\gamma_1^4 \gamma_2^2 \gamma_3^2 (\gamma_2^2 - \gamma_3^2 - 6\gamma_2 + 1)$$

$$0 = 6\gamma_1^4 \gamma_3^3 \varphi_2^2 + (\gamma_3 \varphi_1^2 - 3\gamma_1^4 \gamma_2) \gamma_2^2 \varphi_3^2 - 9\gamma_1^4 \gamma_2 \gamma_3 \varphi_6^2 + 3\gamma_2^3 \gamma_3^3 \varphi_1^2 + 9\gamma_1^4 \gamma_2^2 \gamma_3^2 (\gamma_3^2 - \gamma_2^2 - 6\gamma_3 + 1)$$

(some time passes)

When the dust settles we find **three** backgrounds,  
and no others!

# An approximate background

$$\gamma_1 = 0.22776420155467458$$

$$\gamma_2 = 0.4670546272324634$$

$$\gamma_3 = 0.12728016028858763$$

$$\varphi_1 = 0.14715771499261474$$

$$\varphi_3 = \pm 0.27380714065085027$$

We know little else about this solution, except that it has no additional symmetries.

**Supersymmetry?**

# A circle's worth of backgrounds

$$\gamma_1 = \gamma_2 = \gamma_3 = \frac{1}{3}$$

$$\varphi_1 = -\frac{1}{3}, \quad \varphi_2 = \frac{1}{\sqrt{3}} \cos \alpha, \quad \varphi_3 = \frac{1}{\sqrt{3}} \sin \alpha$$

The 7-dimensional lorentzian geometry is **Sasakian**, and the background has an extra Killing vector: the Sasaki vector field.

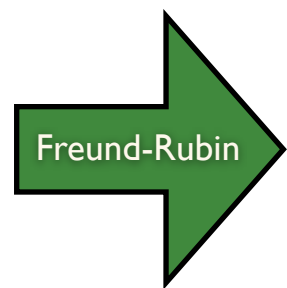
The timelike cone is an 8-dimensional **pseudo-Kähler** manifold; that is, of holonomy  $U(3,1) \subset SO(6,2)$ .

**Supersymmetry?**

# A susy Freund-Rubin background

$$\gamma_2 = \gamma_3 = \frac{2}{3} \quad \gamma_1 = \frac{4}{9} \quad \varphi_1 = \pm \frac{8}{9}$$

The 7-dimensional lorentzian geometry is **Sasaki-Einstein**.  $\therefore$  it has an additional Killing vector and the (time-like) cone is **pseudo-Calabi-Yau**; i.e., of holonomy  $SU(3,1) \subset SO(6,2)$ .



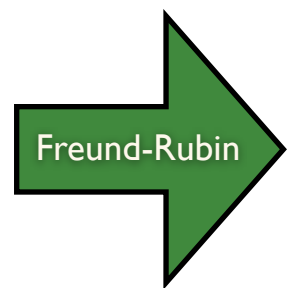
The background is **supersymmetric!**

JMF+Leitner+Simón (20??)

# A susy Freund-Rubin background

$$\gamma_2 = \gamma_3 = \frac{2}{3} \quad \gamma_1 = \frac{4}{9} \quad \varphi_1 = \pm \frac{8}{9}$$

The 7-dimensional lorentzian geometry is **Sasaki-Einstein**.  $\therefore$  it has an additional Killing vector and the (time-like) cone is **pseudo-Calabi-Yau**; i.e., of holonomy  $SU(3,1) \subset SO(6,2)$ .



The background is **supersymmetric!**

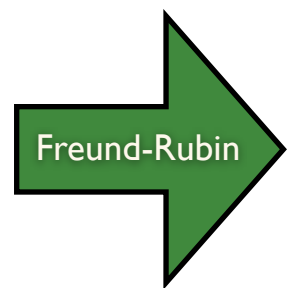
JMF+Leitner+Simón (201?)



# A susy Freund-Rubin background

$$\gamma_2 = \gamma_3 = \frac{2}{3} \quad \gamma_1 = \frac{4}{9} \quad \varphi_1 = \pm \frac{8}{9}$$

The 7-dimensional lorentzian geometry is **Sasaki-Einstein**.  $\therefore$  it has an additional Killing vector and the (time-like) cone is **pseudo-Calabi-Yau**; i.e., of holonomy  $SU(3,1) \subset SO(6,2)$ .



The background is **supersymmetric!**

JMF+Leitner+Simón (201?)

However it is only **N=2** supersymmetric.

# Killing spinors

$$\varepsilon^{Ia} = \zeta^{Ia} \otimes \psi \quad I = 1, \dots, 4 \quad a = 1, 2$$

(symplectic Majorana)

geometric  
Killing  
spinors  
on  $S^4$

geometric  
Killing  
spinor  
on LES

Sasaki  
vector  
field

Killing superalgebra:  $[\varepsilon^{Ia}, \varepsilon^{Jb}] = \Omega^{IJ} \epsilon^{ab} \chi$

$SO(3,2) \times SO(5)$  is an “accidental” symmetry.

(Geometric version of Coleman-Mandula Theorem.)

A LOOK AHEAD

**Still left to do...**

# Still left to do...

➔ **Determine** supersymmetry of remaining backgrounds

## Still left to do...

- ➔ **Determine** supersymmetry of remaining backgrounds
- ➔ **Classify** the  $AdS_4$  backgrounds not of Freund-Rubin type

# Still left to do...

- ➔ **Determine** supersymmetry of remaining backgrounds
- ➔ **Classify** the  $AdS_4$  backgrounds not of Freund-Rubin type
- ➔ **Explore** the homogeneous  $N=4$  candidate backgrounds (most of the work has already been done)

## Still left to do...

- ➔ **Determine** supersymmetry of remaining backgrounds
- ➔ **Classify** the  $AdS_4$  backgrounds not of Freund-Rubin type
- ➔ **Explore** the homogeneous  $N=4$  candidate backgrounds (most of the work has already been done)
- ➔ For completeness: **classify** those cases where  $SO(3,2)$  does not act effectively



# Lessons learnt

# Lessons learnt

✓ Cleaning “corners” is hard work!

# Lessons learnt

- ✓ Cleaning “corners” is hard work!
- ✓ A couple of techniques: “normaliser trick” and determining isometries of  $G/H$

# Lessons learnt

- ✓ Cleaning “corners” is hard work!
- ✓ A couple of techniques: “normaliser trick” and determining isometries of  $G/H$
- ✓ One can “dial up” (a semisimple)  $G$  and hope to classify  $G$ -invariant homogeneous SUGRA backgrounds.

# Lessons learnt

- ✓ Cleaning “corners” is hard work!
- ✓ A couple of techniques: “normaliser trick” and determining isometries of  $G/H$
- ✓ One can “dial up” (a semisimple)  $G$  and hope to classify  $G$ -invariant homogeneous SUGRA backgrounds.
- ✓ It’s fun to interact with bright, motivated UG students. (Thanks to a **Nuffield URB!**)

# Lessons learnt

- ✓ Cleaning “corners” is hard work!
- ✓ A couple of techniques: “normaliser trick” and determining isometries of  $G/H$
- ✓ One can “dial up” (a semisimple)  $G$  and hope to classify  $G$ -invariant homogeneous SUGRA backgrounds.
- ✓ It’s fun to interact with bright, motivated UG students. (Thanks to a **Nuffield URB!**)

*Thank you for your attention!*