From Discrete Integrability to cluster algebras and Back

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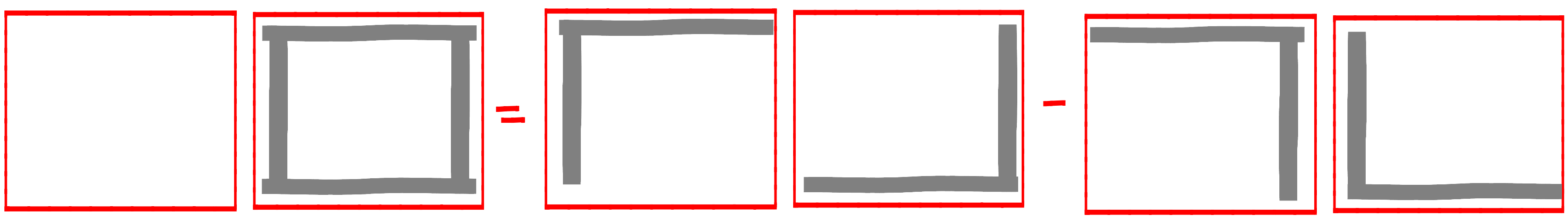
Edinburg, November 2011

An identity for determinants

$M = \alpha \times \alpha$ matrix, $M_{i_1,\dots i_k}^{j_1,\dots j_k}$ = minor with $\left.\begin{matrix}\text{rows } i_1,\dots \\ \text{columns } j_1,\dots\end{matrix}\right\}$ deleted

Relation between Determinants of minors

$$\boxed{|M|\,|M_{1\alpha}^{1\alpha}| = |M_1^1|\,|M_\alpha^\alpha| - |M_1^\alpha|\,|M_\alpha^1|}$$ Desnanot–Jacobi



- an algorithm for computing determinants recursively
- a Plücker relation
- In this talk: a discrete evolution.

Choose $W_{\alpha+1,n} = M = \begin{bmatrix} x_{n-\alpha} & x_{n-\alpha+1} & \cdots & x_{n-1} & x_n \\ x_{n-\alpha+1} & x_{n-\alpha+2} & & x_n & x_{n+1} \\ \vdots & & & & x_{n+2} \\ & & \iddots & & \vdots \\ x_{n-1} & & & & x_{n+\alpha-1} \\ x_n & x_{n+1} & \cdots & x_{n+\alpha-1} & x_{n+\alpha} \end{bmatrix}$ **Discrete Wronskian**

<u>Defn</u>: $\tilde{Q}_{\alpha,n} = \det W_{\alpha,n}$

$$|M|\,|M^{1\alpha}_{1\alpha}| = |M^1_1|\,|M^\alpha_\alpha| - |M^\alpha_1|\,|M^1_\alpha| \Rightarrow$$

$$\boxed{\tilde{Q}_{\alpha+1,n}\,\tilde{Q}_{\alpha-1,n} = \tilde{Q}_{\alpha,n+1}\,\tilde{Q}_{\alpha,n-1} - \tilde{Q}_{\alpha,n}^2}$$

Boundary condition: $\tilde{Q}_{0,n}=1$, $\tilde{Q}_{-1,n}=0$

$\Downarrow$

$$\tilde{Q}_{\alpha,n+1}\,\tilde{Q}_{\alpha,n-1} = \tilde{Q}_{\alpha,n}^2 + \tilde{Q}_{\alpha+1,n}\,\tilde{Q}_{\alpha-1,n}$$

or

$$Q_{\alpha,n+1}\,Q_{\alpha,n-1} = Q_{\alpha,n}^2 - Q_{\alpha+1,n}\,Q_{\alpha-1,n}$$

Q-system

Discrete evolution in n = time

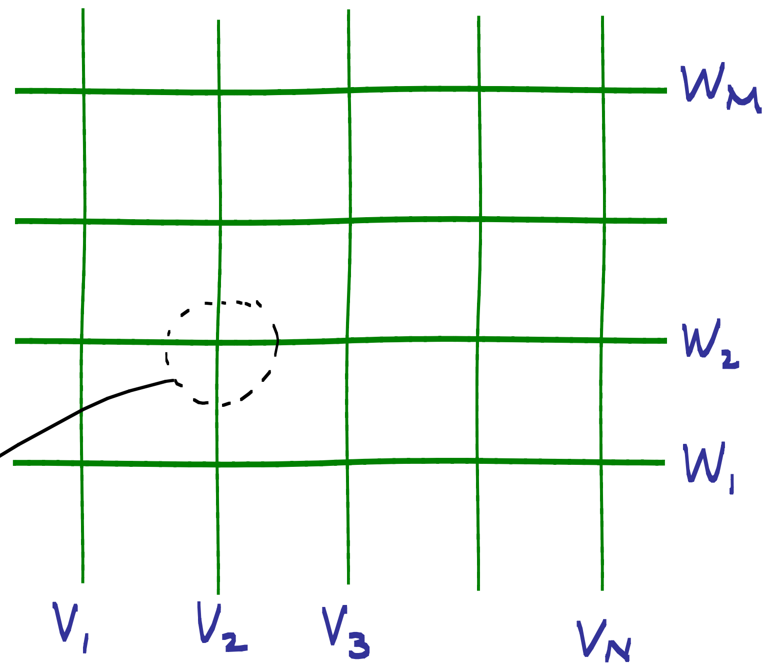
Generalized Heisenberg spin chain:

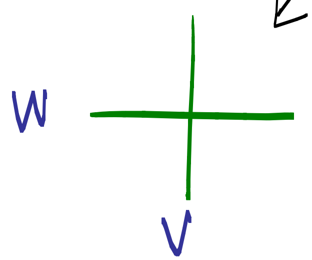
$\{V_i, W_j\}$ = finite-dim vector spaces

Representations of $U_q(\hat{g})$ or $Y(g)$

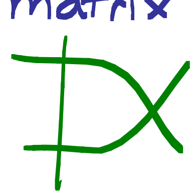
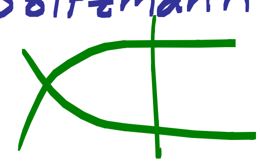
Periodic Boundary conditions

2-D model:





$\rightsquigarrow R_{V,W}$ = matrix of Boltzmann weights

Satisfies  =  Yang-Baxter eq.

$\Rightarrow$ T_W = 

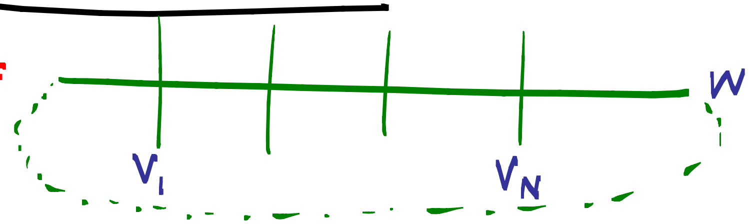
$= \text{Trace}_W(R_{WV_1} R_{WV_2} \cdots R_{WV_N})$

transfer matrix satisfies

$[T_{W_1}, T_{W_2}] = 0$

Integrable

Combinatorics

$\Rightarrow$ $T_W =$ 

$[T_{W_1}, T_{W_2}] = 0$

$$\boxed{T_W \text{ is an operator on } V_1 \otimes \cdots \otimes V_N}$$

$W, V_1, \ldots, V_N$ = Representations of $Y(\mathfrak{g})$ or $U_q(\hat{\mathfrak{g}})$

Example: $W \simeq V_1 \simeq \cdots \simeq V_N \simeq \mathbb{C}^2(z)$ the 2-dimensional rep of $Y(sl_2)$

$R_{W,V}$ = rational $\Rightarrow$ $T \rightsquigarrow$ XXX Hamiltonian

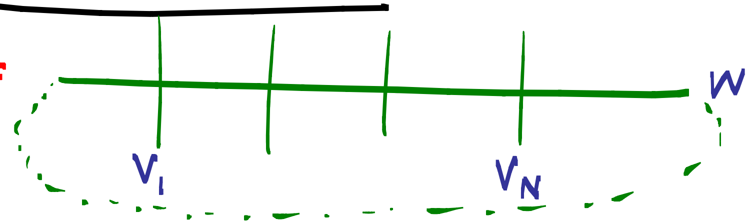
if we choose $U_q(\hat{sl}_2)$ = R-matrix is trigonometric $\rightsquigarrow$ XXZ.

Fact from 80's:

There is a Bethe ansatz solution for generalized XXX / XXZ

if $W, V_1, \ldots, V_N$ are "special" representations

Combinatorics

$\Rightarrow$ $T_W =$ 

$[T_{W_1}, T_{W_2}] = 0$

$\boxed{T_W \text{ is an operator on } V_1 \otimes \cdots \otimes V_N}$

Functional Relations for $\{T_W\}$

"Special" means W, V_i have the form $V_{n\omega_\alpha}(\zeta)$

positive integer

fundamental weight $\omega_1, \dots, \omega_r$

complex number "spectral parameter"

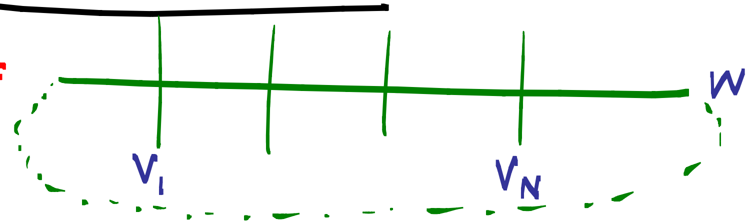
$$T_W = T_{\alpha,n}(\zeta)$$

$$\boxed{T_{\alpha,n+1}(\zeta)\, T_{\alpha,n-1}(\zeta) = T_{\alpha,n}(\zeta+1)\, T_{\alpha,n}(\zeta-1) - \prod_{\beta \sim \alpha} T_{\beta,n}(\zeta)}$$

Fusion relation for T-system

Combinatorics

$\Rightarrow$ $T_W =$



$[T_{W_1}, T_{W_2}] = 0$

$$\boxed{T_W \text{ is an operator on } V_1 \otimes \cdots \otimes V_N}$$

Functional Relations for $\{T_W\}$

$$\boxed{T_{\alpha,n+1}(\zeta)\, T_{\alpha,n-1}(\zeta) = T_{\alpha,n}(\zeta+1)\, T_{\alpha,n}(\zeta-1) - \prod_{\beta\sim\alpha} T_{\beta,n}(\zeta)}$$

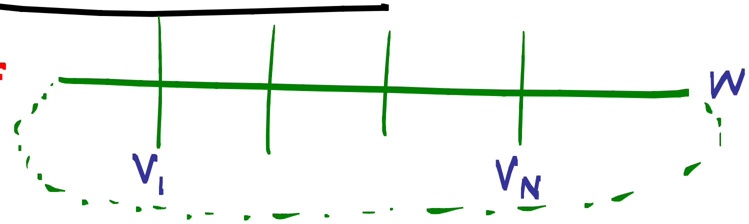
Fusion relation for T-system

$\zeta \to \infty \quad T_{\alpha,n}(\zeta) \to Q_{\alpha,n}$

$$\boxed{Q_{\alpha,n+1}\, Q_{\alpha,n-1} = Q_{\alpha,n}^2 - \prod_{\beta\sim\alpha} Q_{\beta,n}}$$

Same as DT for Discrete Wronskian if $\mathfrak{g} = A_r$

Combinatorics

$\Rightarrow$ $T_W =$ 

$[T_{w_1}, T_{w_2}] = 0$

$$\boxed{T_W \text{ is an operator on } V_1 \otimes \cdots \otimes V_N}$$

Combinatorial question: Does the Bethe ansatz give a complete set of solutions?

Hilbert space $\mathcal{H} = V_1 \otimes \cdots \otimes V_N \underset{\mathfrak{g}\text{-mod}}{\simeq} \bigoplus_\lambda M_{\lambda, \{V_i\}} V_\lambda$

multiplicity

Claim: Combinatorics is governed by the Q-system

$$\boxed{Q_{\alpha,n+1} Q_{\alpha,n-1} = Q_{\alpha,n}^2 - \prod_{\beta \sim \alpha} Q_{\beta,n}}$$

Extra boundaries

$Q_{\alpha,0} = 1$

$Q_{r+1,n} = 1$

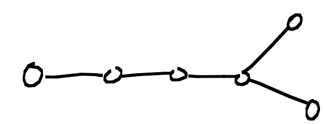
<u>Combinatorial question</u>: Does the Bethe ansatz give a <u>complete</u> set of solutions?

Hilbert space $\mathcal{H} = V_1 \otimes \cdots \otimes V_N \underset{\mathfrak{g}\text{-mod}}{\cong} \bigoplus_{\lambda} M_{\lambda, \{V_i\}} V_\lambda$

multiplicity

<u>Claim</u>: Combinatorics is governed by the Q-system

$$\boxed{Q_{\alpha,n+1}\, Q_{\alpha,n-1} = Q_{\alpha,n}^2 - \prod_{\beta \sim \alpha} Q_{\beta,n}}$$



<u>Claim 2</u>: The linearized spectrum ($\approx$ CFT characters) is governed by the quantum Q-system.

(*) Bethe ansatz solutions are not all eigenvectors but combinatorics okay

(**) Todays slides limited to $\mathfrak{g}$ simply-laced.

$$\boxed{Q_{\alpha,n+1}\,Q_{\alpha,n-1} = Q_{\alpha,n}^2 - \prod_{\beta\sim\alpha} Q_{\beta,n}}$$

Example: $\mathfrak{sl}_2$ $\quad \alpha = 1$ only

$$\boxed{Q_{n+1}\,Q_{n-1} = Q_n^2 - 1} \qquad Q_0 = 1$$

$\Rightarrow Q_n(t)$ $(n \geq 1)$ = chebyshev poly of 2^{nd} kind in Q_1

Example: $\mathfrak{sl}_{r+1}$

$$Q_{\alpha,n+1}\,Q_{\alpha,n-1} = Q_{\alpha,n}^2 - Q_{\alpha+1,n}\,Q_{\alpha-1,n} \;;\quad Q_{0,n} = Q_{r+1,n} = 1$$

$$Q_{\alpha,0} = 1$$

1) If $Q_{\alpha,1} = S_{\left.\begin{smallmatrix}\square\\ \vdots\\ \square\end{smallmatrix}\right\}\alpha} \Rightarrow Q_{\alpha,n}$ = schur functions

2) Characters of the modules $V_{n\omega_\alpha} = V_{\left.\underbrace{\begin{smallmatrix}\square & \cdots & \square\\ \vdots & & \vdots\\ \square & \cdots & \square\end{smallmatrix}}_{n}\right\}\alpha}$ satisfy Q-sys.

Completeness question

$$\mathcal{H} = V_1 \otimes \cdots \otimes V_N \underset{\mathfrak{g}\text{-mod}}{\simeq} \bigoplus_{\lambda} M_{\lambda,\{V_i\}} V_\lambda$$

Bethe ansatz says for sl_2

$$\boxed{M_{l\omega_1,\{V_{\omega_1}^{\otimes n_1}, V_{2\omega_1}^{\otimes n_2},\ldots\}} = \sum_{\substack{m_1,m_2,\ldots \geq 0 \\ p_i \geq 0 \\ \sum i n_i - 2\sum i m_i = l}}' \prod_{i \geq 1} \binom{p_i + m_i}{m_i}}$$

where $p_i = \sum_j \min(i,j)(n_j - 2m_j)$

Completeness: Is $\left|\mathrm{Hom}_{sl_2}\left(V_{l\omega_1}, V_{\omega_1}^{\otimes n_1} \otimes V_{2\omega_1}^{\otimes n_2} \otimes \cdots\right)\right| = M$ above?

Theorem: Yes, if characters of $V_{n\omega_a}$ satisfy Q-system.

To prove completeness theorem we need:

Theorem: Solutions of the equation

$$\boxed{Q_{\alpha,n+1}\, Q_{\alpha,n-1} = Q_{\alpha,n}^2 - \prod_{\beta\sim\alpha} Q_{\beta,n}}$$

with $Q_{\beta,0} = 1$ for all β are <u>polynomials</u> in $\{Q_{\alpha,1}\}_{\alpha=1}^{r}$

Check: It is not obvious that solutions of

$$Q_{n+1} = \frac{Q_n^2 - 1}{Q_{n-1}}$$

are polynomials!!

$$(Q_0, Q_1) \longrightarrow Q_2 = \left.\frac{Q_1^2 - 1}{Q_0}\right|_{Q_0=1} = Q_1^2 - 1 \longrightarrow Q_3 = \frac{Q_2^2 - 1}{Q_1} = \frac{Q_1^4 - 2Q_1^2}{Q_0 Q_1} = \frac{Q_1^3 - 2Q_1}{Q_0}$$

$$\longrightarrow Q_4 = \left.\frac{Q_3^2 - 1}{Q_2}\right|_{Q_0=1} = \text{polynomial in } Q_1$$

Why not just rational functions in (Q_0, Q_1)?

Theorem: Solutions of the equation

$$\boxed{Q_{\alpha,n+1}\, Q_{\alpha,n-1} = Q_{\alpha,n}^2 - \prod_{\beta\sim\alpha} Q_{\beta,n}}$$

with $Q_{\beta,0}=1$ for all β are polynomials in $\{Q_{\alpha,1}\}_{\alpha=1}^{r}$

Why not just rational functions in (Q_0, Q_1)?

Because:

Theorem 1 [RK07] Q-system equations are cluster algebra mutations
$\Rightarrow Q_{\alpha,n}$ are cluster variables.

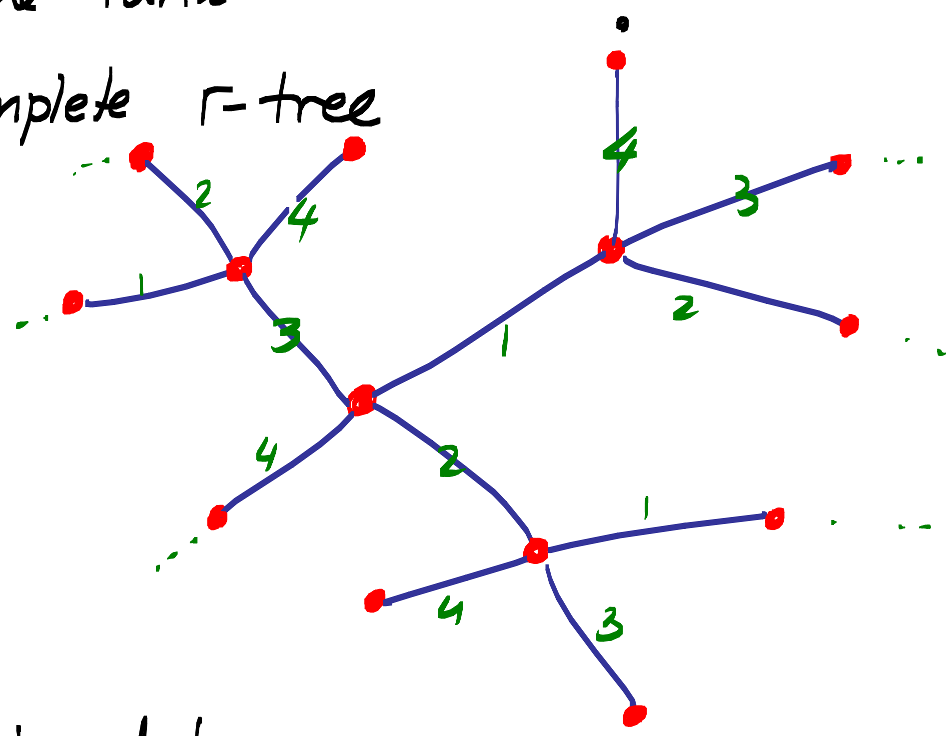
Theorem 2 [Fomin+Zelevinsky, 01] Cluster variables are Laurent polynomials in any initial data [cluster seed].

Corollary: For the Q-system this implies polynomiality.

Cluster algebras

Geometric type
no coefficients...

- $r \in \mathbb{N}$ the rank
- $\mathbb{T}_r$ = complete r-tree



Discrete evolution on a complete r-Tree

Node $\circ$ carries data:

$\vec{x} = (x_1, \ldots, x_r)$

B = integer skew-symmetric matrix

Edge $(x, B) \overset{k}{\text{———}} (x', B') = \mu_k(x, B)$ denotes a "mutation"

= Discrete evolution

Cluster algebras

Node ● carries data:

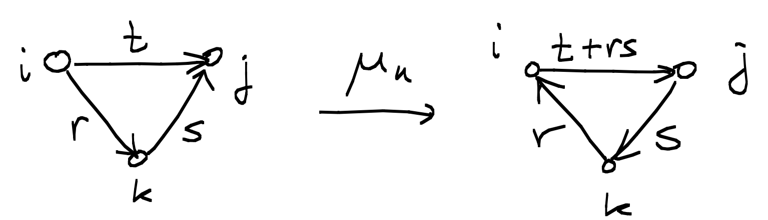
$$\boxed{\begin{array}{l}\vec{x} = (x_1, \dots, x_r) \\ B = \text{integer skew-symmetric matrix}\end{array}}$$

Edge $(x,B) \overset{k}{\circ\!\!-\!\!-\!\!-\!\!\circ} (x',B') = \mu_k(x,B)$ denotes a "mutation"
= Discrete evolution

evolution determined by B:

$$\begin{pmatrix} x_1 \\ \vdots \\ x_r \end{pmatrix} \xrightarrow{\mu_k} \begin{pmatrix} x_1 \\ \vdots \\ x_k' \\ \vdots \\ x_r \end{pmatrix} \qquad x_k' = \frac{\prod_i x_i^{[B_{ik}]_+} + \prod_i x_i^{[-B_{ik}]_+}}{x_k}$$

$$B \xrightarrow{\mu_k} B'$$



Example $B = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = \underset{1}{\circ} \longrightarrow \underset{2}{\circ}$

$$\cdots \circ \overset{}{\underset{2}{\text{———}}} \circ \underset{1}{\text{———}} \circ \underset{1}{\text{———}} \circ \underset{2}{\text{———}} \circ \underset{1}{\text{———}} \circ \underset{2}{\text{———}} \circ \cdots$$

$$\overset{\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}}{\begin{pmatrix} x \\ y \end{pmatrix}} \overset{1}{\text{———}} \overset{\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}}{\begin{pmatrix} \frac{y+1}{x} \\ y \end{pmatrix}} \overset{2}{\text{———}} \overset{\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}}{\begin{pmatrix} \frac{y+1}{x} \\ \frac{x+y+1}{xy} \end{pmatrix}} \overset{1}{\text{———}} \overset{\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}}{\begin{pmatrix} \frac{\frac{x+y+1}{xy}+1}{(y+1)/x} = \frac{x+1}{y} \\ \frac{x+y+1}{xy} \end{pmatrix}}$$

$$\overset{2}{\text{———}} \underset{\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}}{\begin{pmatrix} \frac{x+1}{y} \\ \frac{(x+1+y)\,xy}{y(x+1+y)} = x \end{pmatrix}} \overset{1}{\text{———}} \underset{\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}}{\begin{pmatrix} \frac{(x+1)}{(x+1)/y} = y \\ x \end{pmatrix}} = \begin{pmatrix} y \\ x \end{pmatrix} \overset{\sigma}{\longleftrightarrow} \begin{pmatrix} x \\ y \end{pmatrix}$$

Periodic

Theorem: Finite cluster algebras $\longleftrightarrow$ finite Lie algebras

Example $\quad B = \begin{pmatrix} 0 & 2 \\ -2 & 0 \end{pmatrix} = \underset{1}{\circ} \xrightarrow{\ 2\ } \underset{2}{\circ}$

$$\cdots \circ \overset{}{\underset{2}{—}} \circ \underset{1}{—} \circ \underset{1}{—} \circ \underset{2}{—} \circ \underset{1}{—} \circ \underset{2}{—} \circ \cdots$$

$$\underset{\begin{pmatrix} 0 & 2 \\ -2 & 0 \end{pmatrix}}{\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}} \underset{1}{—} \underset{\begin{pmatrix} 0 & -2 \\ 2 & 0 \end{pmatrix}}{\begin{pmatrix} \frac{x_2^2+1}{x_1} \\ x_2 \end{pmatrix}} \equiv \begin{pmatrix} x_3 \\ x_2 \end{pmatrix} \underset{2}{—} \underset{\begin{pmatrix} 0 & 2 \\ -2 & 0 \end{pmatrix}}{\begin{pmatrix} x_3 \\ \frac{x_3^2+1}{x_2} \end{pmatrix}} \equiv \begin{pmatrix} x_3 \\ x_4 \end{pmatrix} — \cdots$$

This is the renormalized A_1 Q-system

$$x_{n+1} = \frac{x_n^2+1}{x_{n-1}}$$

Theorem: The Q-system equations are mutations in the cluster algebra containing the seed

$$X = (\tilde{Q}_{1,0}, \tilde{Q}_{2,0}, \dots; \tilde{Q}_{1,1}, \dots, \tilde{Q}_{r,1}) \quad \text{and} \quad B = \left(\begin{array}{c|c} 0 & -C \\ \hline C & 0 \end{array}\right)$$

About cluster algebras

It is known that any cluster variable is a Laurent polynomial (not just a rational function!) with integer coefficients in terms of the cluster seed at any other node (that is, any initial data)

Conjecture: The coefficients of the Laurent polynomial are non-negative [Fomin, Zelevinsky 2000]

Proven case by case - still open in general.

About cluster algebras

It is known that any cluster variable is a **Laurent polynomial** (not just a rational function!)

Example of how to use this:

This implies polynomiality of Q_n in Q_1 in $Q_{n+1} = \frac{Q_n^2 - 1}{Q_{n-1}}$ after $Q_0 \mapsto 1$

$$Q_n = \sum_{i=-a}^{b} P_i(Q_0)\, Q_1^{i} = \sum_{i=0}^{b} P_i(Q_0)\, Q_1^{i} + \sum_{0 < i \leq a} P_{-i}(Q_0)\, Q_1^{-i}$$

$$P_{-i}(Q_0)\, Q_1^{-i} = P_{-i}(Q_0) \left(\frac{Q_0^2 - 1}{Q_{-1}} \right)^{-i} = \frac{Q_{-1}^{i}\, P_{-i}(Q_0)}{(Q_0^2 - 1)^{i}}$$ $\leftarrow$ rational

$P_{-i}(Q_0)$ must be divisible by $(Q_0^2 - 1)^i$

$$\Rightarrow P_{-i}(Q_0)\Big|_{Q_0 = 1} = 0 \text{ if } i > 0 \quad \Rightarrow \text{QED.}$$

About cluster algebras

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This implies polynomiality of Q_n in Q_1 in $Q_{n+1} = \frac{Q_n^2 - 1}{Q_{n-1}}$ after $Q_0 \mapsto 1$

For the Q-systems this is generalized to:

Thm: Any cluster variable in the Q-system cluster algebra is a **Polynomial** in $\{Q_{1,1}, \ldots, Q_{1,r}\}$ under the boundary condition $\{Q_{\alpha,-1} = 0\}_{\alpha=1,\ldots,r}$

(Representation-theoretical b.c.)

$$\boxed{Q_{\alpha,n+1}\, Q_{\alpha,n-1} = Q_{\alpha,n}^2 - \prod_{\beta\sim\alpha} Q_{\beta,n}}$$

[From now on $\mathfrak{g}$ = mostly sl_{r+1}]

Claim: The Q-system is a discrete integrable system in discrete time n.

Integrable means: 1) There are r "integrals of the motion"

= algebraically independent functions of $\{Q_{\alpha,n}\}$ which are independent of n.

2) $\{Q_{1,n}\}$ satisfy a linear recursion relation with constant coefficients (IOM)

$\Rightarrow$ solvable.

Example of "integrable" and "solvable"

$\boxed{sl_2}$ $$\tilde{Q}_{n+1}\tilde{Q}_{n-1} = \tilde{Q}_n^2 + 1$$

$$1 = \tilde{Q}_{n+1}\tilde{Q}_{n-1} - \tilde{Q}_n^2 = \left|\begin{pmatrix} Q_{n-1} & \tilde{Q}_n \\ \tilde{Q}_n & \tilde{Q}_{n+1} \end{pmatrix}\right|$$

Example of "integrable" and "solvable"

$\boxed{sl_2}$ $\tilde{Q}_{n+1}\tilde{Q}_{n-1} = \tilde{Q}_n^2 + 1$

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$$= \left|\begin{pmatrix} \tilde{Q}_n & \tilde{Q}_{n+1} \\ \tilde{Q}_{n+1} & \tilde{Q}_{n+2} \end{pmatrix}\right| = -\left|\begin{pmatrix} \tilde{Q}_{n+1} & \tilde{Q}_n \\ \tilde{Q}_{n+2} & \tilde{Q}_{n+1} \end{pmatrix}\right|$$

same equation with $n \mapsto n+1$

switch columns

Example of "integrable" and "solvable"

$\boxed{sl_2}$ $\quad \tilde{Q}_{n+1}\tilde{Q}_{n-1} = \tilde{Q}_n^2 + 1$

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$$= \left|\begin{pmatrix} \tilde{Q}_n & \tilde{Q}_{n+1} \\ \tilde{Q}_{n+1} & \tilde{Q}_{n+2} \end{pmatrix}\right| = - \left|\begin{pmatrix} \tilde{Q}_{n+1} & \tilde{Q}_n \\ \tilde{Q}_{n+2} & \tilde{Q}_{n+1} \end{pmatrix}\right|$$

subtract
1-1

$$\Rightarrow \left|\begin{pmatrix} \tilde{Q}_{n-1} + \tilde{Q}_{n+1} & \tilde{Q}_n \\ \tilde{Q}_n + \tilde{Q}_{n+2} & \tilde{Q}_{n+1} \end{pmatrix}\right| = 0$$

Example of "integrable" and "solvable"

$\boxed{sl_2}$ $\quad \tilde{Q}_{n+1}\tilde{Q}_{n-1} = \tilde{Q}_n^2 + 1$

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$$= \left|\begin{pmatrix} \tilde{Q}_n & \tilde{Q}_{n+1} \\ \tilde{Q}_{n+1} & \tilde{Q}_{n+2} \end{pmatrix}\right| = -\left|\begin{pmatrix} \tilde{Q}_{n+1} & \tilde{Q}_n \\ \tilde{Q}_{n+2} & \tilde{Q}_{n+1} \end{pmatrix}\right|$$

$$\Rightarrow \left|\begin{pmatrix} \tilde{Q}_{n-1} + \tilde{Q}_{n+1} & \tilde{Q}_n \\ \tilde{Q}_n + \tilde{Q}_{n+2} & \tilde{Q}_{n+1} \end{pmatrix}\right| = 0$$

$\Rightarrow \tilde{Q}_{n-1} + \tilde{Q}_{n+1} = c\tilde{Q}_n$

& $\tilde{Q}_n + \tilde{Q}_{n+2} = c\tilde{Q}_{n+1}$ $\Rightarrow$ $\boxed{c = \dfrac{\tilde{Q}_{n-1} + \tilde{Q}_{n+1}}{\tilde{Q}_n}}$ is independent of n

IOM

$\tilde{Q}_{n+1} - c\tilde{Q}_n + \tilde{Q}_{n-1} = 0$ ← linear equation

Example of "integrable" and "solvable"

$\boxed{sl_2}$ $\quad \tilde{Q}_{n+1}\tilde{Q}_{n-1} = \tilde{Q}_n^2 + 1$

$$1 = \tilde{Q}_{n+1}\tilde{Q}_{n-1} - \tilde{Q}_n^2 = \left|\begin{pmatrix} \tilde{Q}_{n-1} & \tilde{Q}_n \\ \tilde{Q}_n & \tilde{Q}_{n+1} \end{pmatrix}\right|$$

$$= \left|\begin{pmatrix} \tilde{Q}_n & \tilde{Q}_{n+1} \\ \tilde{Q}_{n+1} & \tilde{Q}_{n+2} \end{pmatrix}\right| = -\left|\begin{pmatrix} \tilde{Q}_{n+1} & \tilde{Q}_n \\ \tilde{Q}_{n+2} & \tilde{Q}_{n+1} \end{pmatrix}\right|$$

$$\tilde{Q}_{n+1} - c\tilde{Q}_n + \tilde{Q}_{n-1} = 0$$

In general, linear EOM comes from

$$0 = \left|\begin{pmatrix} \tilde{Q}_{n-r-1} & \cdots & \tilde{Q}_n \\ \vdots & \iddots & \\ \tilde{Q}_n & & \tilde{Q}_{n+r+1} \end{pmatrix}\right|_{(r+1)\times(r+1)} = W_n^{(r+1)}$$

Expand determinant

IOM = determinants of minors

Example of "integrable" and "solvable"

$$\tilde{Q}_{n+1}\tilde{Q}_{n-1} = \tilde{Q}_n^2 + 1$$

$$\Downarrow$$

$$\tilde{Q}_{n+1} - c\tilde{Q}_n + \tilde{Q}_{n-1} = 0$$

$C = \dfrac{\tilde{Q}_{n-1}+\tilde{Q}_{n+1}}{\tilde{Q}_n}$ is independent of n

$$\Downarrow$$

$$C = \frac{\tilde{Q}_0}{Q_1} + \frac{\tilde{Q}_2}{Q_1} = \frac{\tilde{Q}_0}{Q_1} + \frac{\tilde{Q}_1^2+1}{\tilde{Q}_1\tilde{Q}_0}$$

$$= \underbrace{\frac{\tilde{Q}_0}{\tilde{Q}_1}}_{y_3} + \underbrace{\frac{\tilde{Q}_1}{\tilde{Q}_0}}_{y_1} + \underbrace{\frac{1}{\tilde{Q}_0\tilde{Q}_1}}_{y_2}$$

"weights"

$y_1 y_3 = 1$

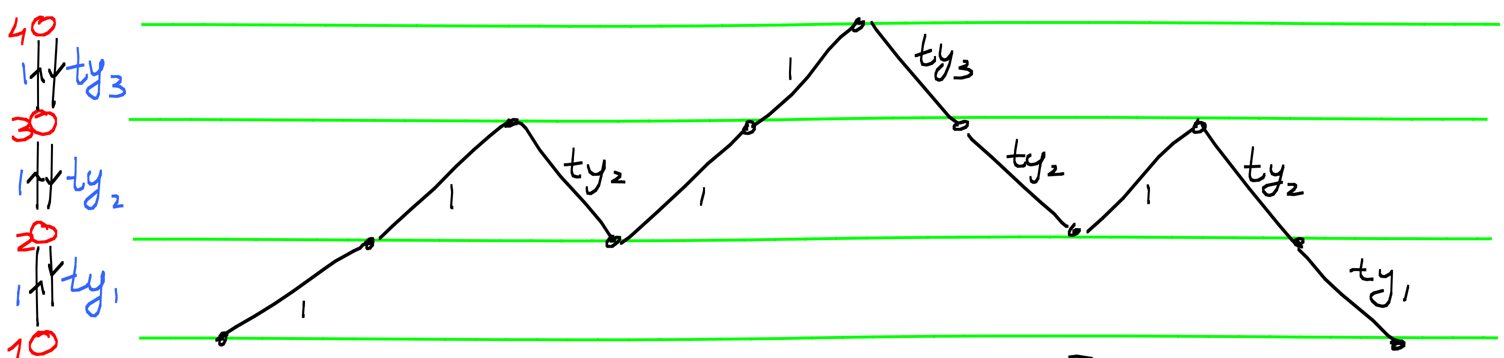
Defn: $Q(t) = \sum_{n\geq 0} \tilde{Q}_n t^n$

$$\frac{Q(t)}{\tilde{Q}_0} = \cfrac{1}{1 - \cfrac{ty_1}{1 - \cfrac{ty_2}{1 - ty_3}}}$$

Path model

$$\frac{Q(t)}{Q_0} = \cfrac{1}{1 - \cfrac{ty_1}{1 - \cfrac{ty_2}{1 - ty_3}}} = \sum_{n\geq 0} \frac{\tilde{Q}_n}{Q_0} t^n \Rightarrow$$

$\frac{Q_n}{Q_0}$ = partition function of paths of length $2n$ from vertex 1 to itself on the weighted graph

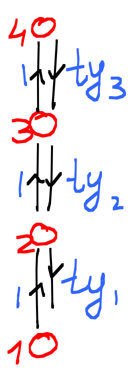


weight of this path: $t^5 y_2^3 y_1 y_3$

Path model

$$\frac{Q(t)}{Q_0} = \cfrac{1}{1 - \cfrac{ty_1}{1 - \cfrac{ty_2}{1 - ty_3}}} = \sum_{n \geq 0} \frac{\tilde{Q}_n t^n}{Q_0} \Rightarrow$$

$\frac{Q_n}{Q_0}$ = partition function of paths of length $2n$ from vertex 1 to itself



Proof: T = transfer matrix

$[T]_{ij}$ = weight of step from j to i

$$T = \begin{pmatrix} 0 & ty_1 & 0 & 0 \\ 1 & 0 & ty_2 & 0 \\ 0 & 1 & 0 & ty_3 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

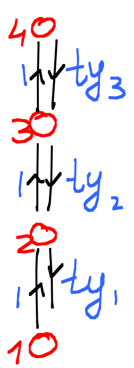
$[T^n]_{11}$ = partition function of paths of length n from 1 to 1

$$\Rightarrow \frac{Q(t)}{Q_0} = \sum_{n \geq 0} [T^n]_{11} = [(\mathbb{1} - T)^{-1}]_{11} = \text{continued fraction above.}$$

Path model

$$\frac{Q(t)}{\tilde{Q}_0} = \cfrac{1}{1 - \cfrac{ty_1}{1 - \cfrac{ty_2}{1 - ty_3}}} = \sum_{n \geq 0} \frac{\tilde{Q}_n}{Q_0} t^n \Rightarrow$$

$\frac{Q_n}{Q_0}$ = partition function of paths of length $2n$ from vertex 1 to itself



$$y_3 = \frac{\tilde{Q}_0}{Q_1}, \quad y_2 = \frac{1}{\tilde{Q}_0 \tilde{Q}_1}, \quad y_1 = \frac{\tilde{Q}_1}{Q_0}$$

positive monomials in initial data

$\Rightarrow$ $\tilde{Q}_n$ - positive Laurent polynomial with integer coefficients in initial data

+ translational invariance $\Rightarrow$ Laurent positivity prop.

A_2 Q-system cluster algebra

$\tilde{Q}_{1,n} \equiv R_n$, $\tilde{Q}_{2,n} \equiv P_n$

$$\begin{cases} R_{n+1}R_{n-1} = R_n^2 + P_n \\ P_{n+1}P_{n-1} = P_n^2 + R_n \end{cases}$$

Integrable

there are 2 integrals of motion and a linear recursion relation.

$$W_{\alpha,n} = \begin{vmatrix} R_{n-\alpha+1} & \cdots & R_n \\ \vdots & \iddots & \vdots \\ R_n & \cdots & R_{n+\alpha-1} \end{vmatrix} : \quad W_{1,n} = R_n,$$

$$W_{2,n} = P_n = \begin{vmatrix} R_{n-1} & R_n \\ R_n & R_{n+1} \end{vmatrix}$$

Boundary conditions: $W_{0,n} \equiv 1$

$W_{3,n} = \tilde{Q}_{3,n} \equiv 1$ for $\mathfrak{g} = \mathfrak{sl}_3$ $\Rightarrow W_{3,n} - W_{3,n-1} = 0$

$\Rightarrow$ Linear recursion + IOM.

A_2 Q-system cluster algebra

$\tilde{Q}_{1,n} \equiv R_n, \quad \tilde{Q}_{2,n} \equiv P_n$

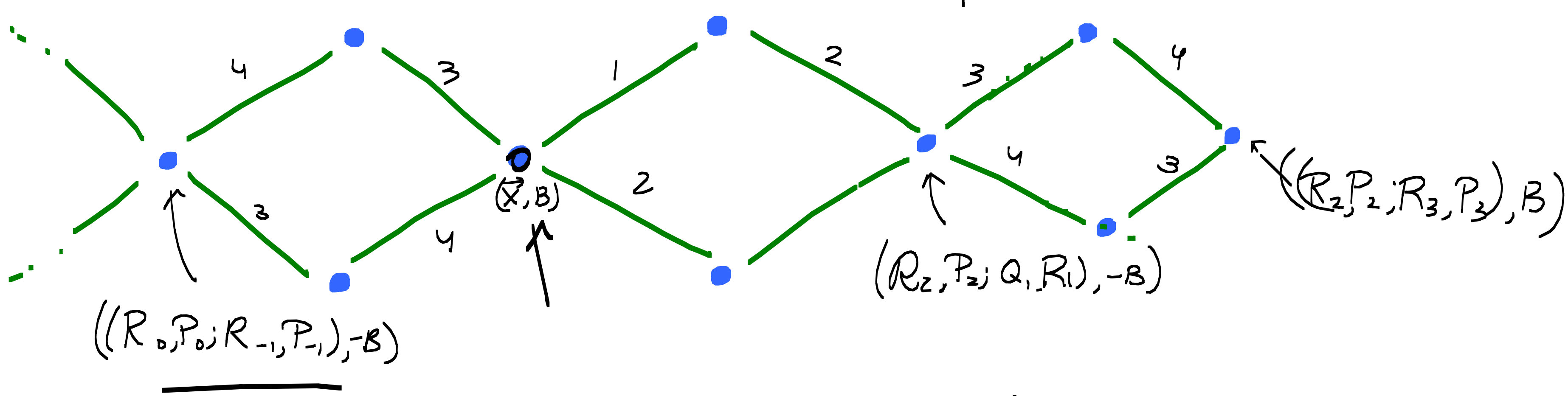
$$\begin{cases} R_{n+1}R_{n-1} = R_n^2 + P_n \\ P_{n+1}P_{n-1} = P_n^2 + R_n \end{cases}$$

<u>Integrable</u>

there are <u>2 integrals of motion</u> and a <u>linear recursion relation</u>.

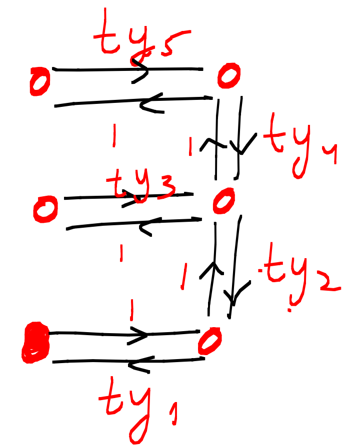
<u>Cluster algebra:</u>

$\vec{x} = (\underset{1}{R_0}, \underset{2}{P_0}; \underset{3}{R_1}, \underset{4}{P_1})$, $\quad B = \underset{4\times 4}{\begin{pmatrix} 0 & C \\ -C & 0 \end{pmatrix}}$, $C = \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}$



Solution of the A_2 Q-system

$R_0^{-1} \sum_{n\geq 0} R_n t^n$ = partition function for paths on:



from node ● to itself.

$y_1, \ldots, y_5$ = + Laurent monomials in (R_0, R_1, P_0, P_1)

e.g. $R_2 = y_1(y_1 + y_2)$, $R_3 = y_1(y_1^2 + y_1 y_2 + y_2 y_3 + y_2 y_4)$

$\ldots$

$$y_{2i-1} = \frac{\tilde{Q}_{i-1,0}\,\tilde{Q}_{i,1}}{\tilde{Q}_{i,0}\,\tilde{Q}_{i-1,1}}, \qquad y_{2i} = \frac{\tilde{Q}_{i-1,0}\,\tilde{Q}_{i+1,1}}{\tilde{Q}_{i,0}\,\tilde{Q}_{i,1}}$$

Solution in terms of cont. fraction:

$$R_0^{-1}R(t) = 1 + \cfrac{ty_1}{1 - ty_1 - \cfrac{ty_2}{1 - ty_3 - \cfrac{ty_4}{1 - ty_6}}}$$

by using

1) linear recursion
2) conserved quantities

Mutations:

1) $$\frac{1}{1 - \frac{a}{1-b}} = 1 + \frac{a}{1-a-b}$$

2) $$a + \frac{b}{1-c} = \frac{a'}{1 - \frac{b'}{1-c'}} \quad \text{if} \quad \begin{cases} a' = a+b \\ b' = bc/a' \\ c' = ac/a' \end{cases}$$

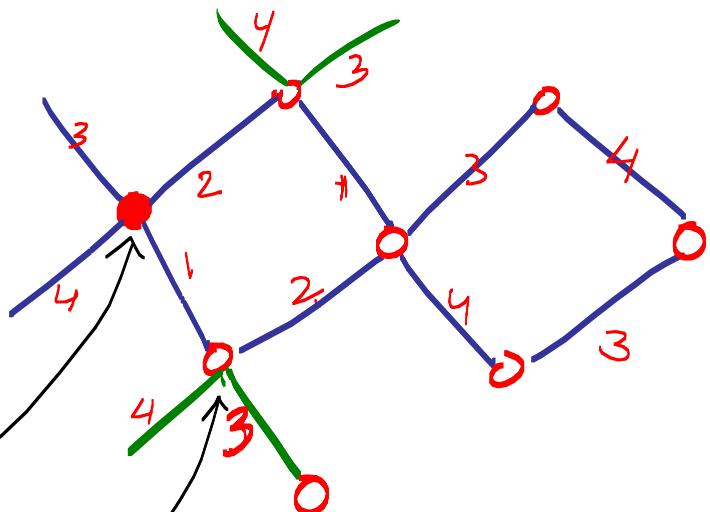
$\Rightarrow$

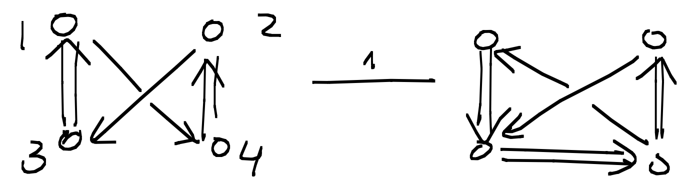
Thm: a mutation acts on weights and graphs to give R_n as p.f. of mutated initial data.

⚠ Not all mutations in our Q-system cluster algebras are Q-system equations!

For $\mathfrak{g} = sl_3$,

$\bullet : B = \left(\begin{array}{cc|cc} 0 & 0 & -2 & 1 \\ 0 & 0 & 1 & -2 \\ \hline 2 & -1 & 0 & 0 \\ -1 & 2 & 0 & 0 \end{array}\right) \qquad X = \left(\begin{array}{c} R_0 \\ P_0 \\ \hline R_1 \\ P_1 \end{array}\right)$





$\left(\begin{array}{cc|cc} 0 & 0 & 2 & -1 \\ 0 & 0 & 1 & -2 \\ \hline -2 & -1 & 0 & 2 \\ 1 & 2 & -2 & 0 \end{array}\right)$

$$\left(\begin{array}{c} R_0 \\ P_0 \\ \hline R_1 \\ P_1 \end{array}\right) \overset{1}{\text{——}} \left(\begin{array}{c} \frac{R_1^2+P_1}{R_0} \\ P_0 \\ \hline R_1 \\ P_1 \end{array}\right) = \left(\begin{array}{c} R_2 \\ P_0 \\ \hline R_1 \\ P_1 \end{array}\right) \overset{3}{\text{——}} \left(\begin{array}{c} - \\ \hline \frac{R_2^2P_0+P_1^2}{R_1} \\ \end{array}\right)$$

← not a Q-system evolution

a mutation in a "non-integrable" direction.

"Valid initial data" of Q-system:

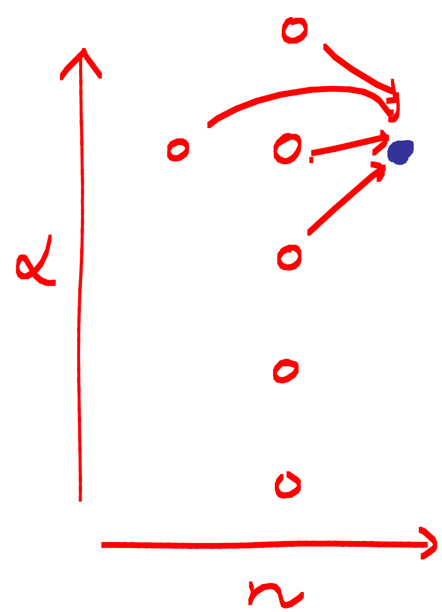
$$\tilde{Q}_{\alpha,n+1} = \frac{\tilde{Q}_{\alpha,n}^2 + \tilde{Q}_{\alpha+1,n}\tilde{Q}_{\alpha-1,n}}{\tilde{Q}_{\alpha,n}}$$

is a function of

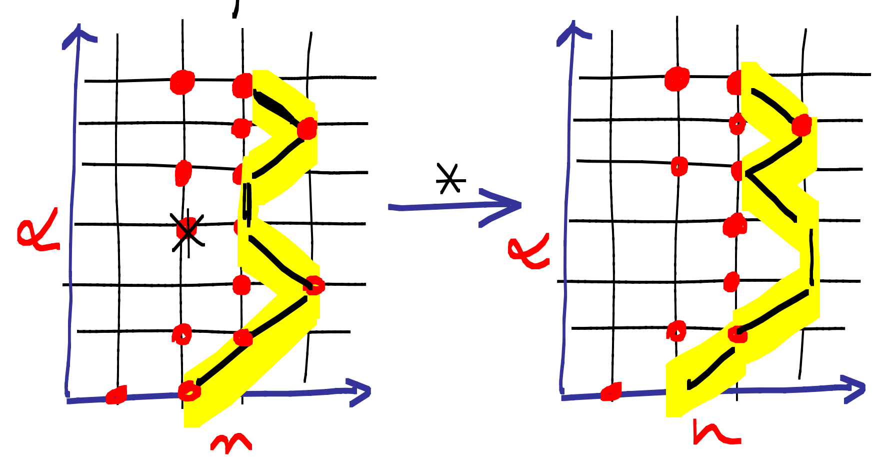
$\tilde{Q}_{\alpha-1,n}$

$\tilde{Q}_{\alpha,n-1}$ $\tilde{Q}_{\alpha,n}$

$\tilde{Q}_{\alpha+1,n}$



$\Rightarrow$ The cluster seeds which correspond to Q-system evolutions correspond to Motzkin paths:



<u>Theorem</u>: Q-systems are discrete integrable evolutions
Solutions satisfy linear recursion relations

- Coefficients = IOM
- Solutions are path partition functions
- Mutations on initial data preserve positivity
  [graph + weights change under mutations]

$\Rightarrow$ Path models $\longrightarrow$ positivity

$\longrightarrow$ explicit form of solutions

$\longrightarrow$ Generalization to non-commutative case ...

Non-commutative Q-system

[Non-commutative wall-crossing formula of Kontsevich]

$$Q_{n+1} Q_{n-1} = Q_n^2 + 1$$

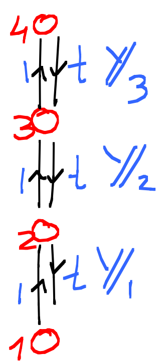
$$\boxed{Q_{n+1} Q_n^{-1} Q_{n-1} = Q_n + Q_n^{-1}}$$

Symplectomorphism:

$K = Q_{n+1}^{-1} Q_n Q_{n+1} Q_n^{-1}$ independent of n

Repeat the analysis above:

Theorem: $Q_n Q_0^{-1}$ is the partition function of paths of length $2n$ from vertex 1 to itself on the graph



$$Y_1 = Q_1 Q_0^{-1}, \quad Y_2 = Q_1^{-1} Q_0^{-1}, \quad Y_3 = Q_1^{-1} Q_0$$

Conserved quantity:

$$C = Q_{n+1} Q_n^{-1} + Q_{n+1}^{-1} Q_n^{-1} + Q_{n+1}^{-1} Q_n$$

Non-commutative Q-system

[Non-commutative wall-crossing formula of Kontsevich]

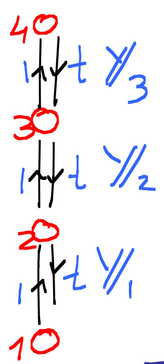
$$Q_{n+1} Q_{n-1} = Q_n^2 + 1$$

$$\boxed{Q_{n+1} Q_n^{-1} Q_{n-1} = Q_n + Q_n^{-1}}$$

$$\boxed{\begin{array}{c}\underline{\text{Symplectomorphism:}} \\ K = Q_{n+1}^{-1} Q_n Q_{n+1} Q_n^{-1} \\ \text{independent of } n\end{array}}$$

Repeat the analysis above:

Theorem: $Q_n Q_0^{-1}$ is the partition function of paths of length $2n$ from vertex 1 to itself on the graph



$$\sum_{n\geq 0} Q_n Q_0^{-1} t^n = \sum_{n\geq 0} [T^n]_{11} = [(\mathbb{1} - T)^{-1}]_{11}$$

$$= \left(1 - t\left[1 - t\left(1 - tY_3\right)^{-1} Y_2\right]^{-1} Y_1\right)^{-1}$$

$$T = \begin{bmatrix} 0 & tY_1 & 0 & 0 \\ 1 & 0 & tY_2 & 0 \\ 0 & 1 & 0 & tY_3 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

Non-commutative rank 2 formula [Non-commutative wall-crossing formula of Kontsevich]

$$T_a : \begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{pmatrix} xyx^{-1} \\ (1+y^a)x^{-1} \end{pmatrix} \qquad K = xyx^{-1}y^{-1} \text{ conserved}$$

take $x_0 = Kx \mapsto (1+x_1^a)x^{-1} = x_2$

$x_1 = y \mapsto Ky = Kx_1$

Then $\boxed{T_a(Kx_n, x_{n+1}) \mapsto (Kx_{n+1}, x_{n+2})}$

Conjecture [Kontsevich]:

1) $x_n \in \mathbb{Z}_+[x_0^{\pm 1}, x_1^{\pm 1}]$

2) coefficients in $\{0,1\}$.

Laurent $\to$ Berenstein-Retakh

Positivity?

Outlook

1) In rank 2, Kontsevich evolution defined for any $B = \begin{pmatrix} 0 & a \\ -b & 0 \end{pmatrix}$ (skew-symmetrizable)

$ab < 4 \Rightarrow Q_n$ quasi-periodic

$ab = 4 \Rightarrow$ Integrable discrete evolution [DFK10]

$a = b$ and $ab > 4$: Path solution by Schiffler-Lee [2011]

$a \neq b$ open

Q: How to define non-commutative evolution for higher rank?

<u>Partial answer:</u> Weights given by Quasi-determinants

mutations = fraction rearrangements give

<u>Non-commutative Hirota equation</u> for Quasi-determinants.

Outlook

2) Quantization:

Choose $$Q_{\alpha,i}\, Q_{\beta,j} = q^{\Lambda^{ij}_{\alpha\beta}}\, Q_{\beta j}\, Q_{\alpha i}$$

if variables are in the same cluster

$\Rightarrow$ Quantum cluster algebra

Solutions: Path p.f. with q-commutative weights.

* Discrete quantum Liouville equation of Faddeev-Volkov

[quantum T-system for $sl_2 \longrightarrow$ quantum Y-system

= evolution of Faddeev-Volkov]

<u>References</u>

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'Rk : arXiv: 0712.2695

2) T-systems as cluster algebras:
Di Francesco, K. : 0803.0362

3) Solutions for A_r Q-systems: 0811.3027
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4) Kotsevich evolution: 0909.0615
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⋮