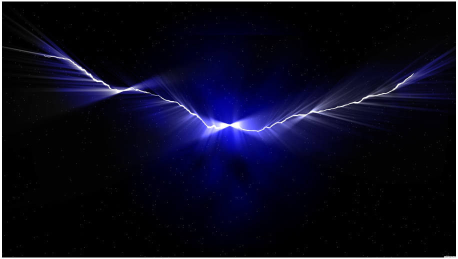
A Fractured XXZ Chain



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Heriot-Watt University, Edinburgh

EMPG, November 2nd, 2011

Plan

1. The Model - Definition & Motivation
2. The Vertex Operator Approach
3. The General Integral Expression
4. Summary & Discussion

Refs:

- Correlation Functions and the Boundary qKZ Equation in a Fractured XXZ Chain, RW: arXiv:1110.2032
- Builds on formalism of 'Algebraic Analysis ...' by Jimbo & Miwa (95), and boundary papers by Jimbo, Kedem, Konno, Kojima/RW, Miwa: hep-th:9411112/9502060

The Model

- Make use of 6V bulk and boundary weights (with $V = \mathbb{C}v_+ \oplus \mathbb{C}v_-$)

$$R(\zeta) = \frac{1}{\kappa(\zeta)} \begin{pmatrix} 1 & & & \\ & \frac{(1-\zeta^2)q}{1-q^2\zeta^2} & \frac{(1-q^2)\zeta}{1-q^2\zeta^2} & \\ & \frac{(1-q^2)\zeta}{1-q^2\zeta^2} & \frac{(1-\zeta^2)q}{1-q^2\zeta^2} & \\ & & & 1 \end{pmatrix} : V \otimes V \to V \otimes V$$

$$K(\zeta; r) = \frac{1}{f(\zeta; r)} \begin{pmatrix} \frac{1-r\zeta^2}{\zeta^2-r} & 0 \\ 0 & 1 \end{pmatrix} : V \to V$$

obeying usual YB, xing and unitarity (for bulk and boundary).

Diagramatics

- Let $K_\bullet(\zeta) = K(\zeta; r)$, and $K_\circ(\zeta) = K(-q^{-1}\zeta^{-1}; r)$. Then

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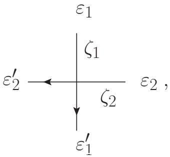
$R^{\varepsilon_1,\varepsilon_2}_{\varepsilon_1',\varepsilon_2'}(\zeta_1/\zeta_2) =$


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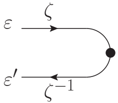

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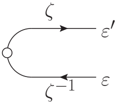

Finite Transfer Matrices

- Let $\mathcal{T}(\zeta) := R_{0N}(\zeta) \cdots R_{02}(\zeta) R_{01}(\zeta) \in \mathrm{End}(V_0 \otimes V_1 \otimes \cdots \otimes V_N)$

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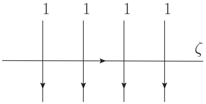
$=$ 

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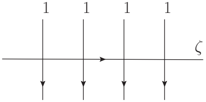
$=$ 

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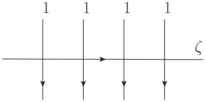
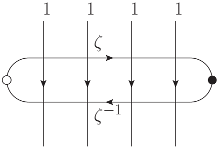
$=$ 

$T^{fin}(\zeta) := \mathrm{Tr}_{V_0}(\mathcal{T}(\zeta))$

- $T_B^{fin}(\zeta) := \mathrm{Tr}_{V_0}(K_\circ(\zeta)\mathcal{T}^{-1}(\zeta^{-1})K_\bullet(\zeta)\mathcal{T}(\zeta)) =$ 

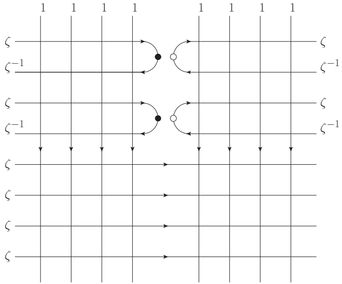
Hamiltonia

- $H^{fin} := \frac{1-q^2}{2q}\frac{d}{d\zeta}\log T^{fin}(\zeta)|_{\zeta=1}$
$$= -\frac{1}{2}\sum_{n=1}^{N}\left(\sigma^x_{i+1}\sigma^x_i + \sigma^y_{i+1}\sigma^y_i + \Delta\sigma^z_{i+1}\sigma^z_i\right) + const$$
where $\Delta = (q+q^{-1})/2$

- $H^{fin}_B := \frac{1-q^2}{4q}\frac{d}{d\zeta}T^{fin}_B(\zeta)|_{\zeta=1}$
$$= -\frac{1}{2}\sum_{n=1}^{N-1}\left(\sigma^x_{i+1}\sigma^x_i + \sigma^y_{i+1}\sigma^y_i + \Delta\sigma^z_{i+1}\sigma^z_i\right) + h\sigma^z_1 - h\sigma^z_N + const$$
where $h = \frac{(q-q^{-1})}{4}\frac{1+r}{1-r}$

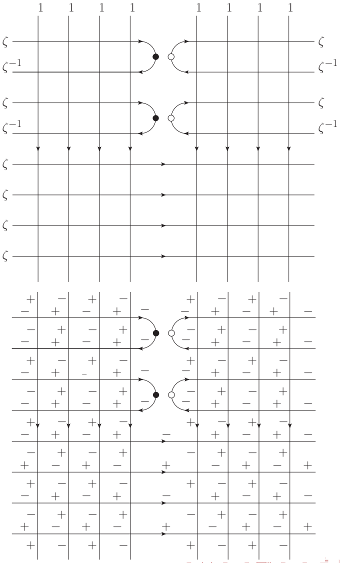
Infinite Partition Function

- Interested in



Infinite Partition Function

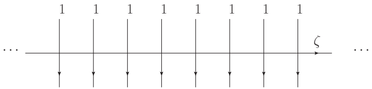
- Interested in
- with antiferromagnetic BC for $i = 0$



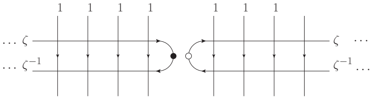
Infinite Transfer Matrices

- Hence, concerned with two transfer matrices:

Bulk $T(\zeta) =$



Fracture $T'(\zeta) =$



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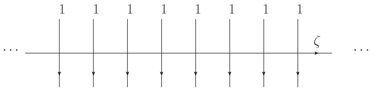

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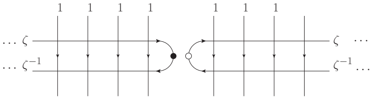

- corresponding to

$$H = -\frac{1}{2}\sum_{n\in\mathbb{Z}}\left(\sigma^x_{i+1}\sigma^x_i + \sigma^y_{i+1}\sigma^y_i + \Delta\sigma^z_{i+1}\sigma^z_i\right),$$

$$H' = H_L + H_R, \quad H_{L,R},$$

with $$H_L = -\frac{1}{2}\sum_{n\geq 1}\left(\sigma^x_{i+1}\sigma^x_i + \sigma^y_{i+1}\sigma^y_i + \Delta\sigma^z_{i+1}\sigma^z_i\right) + h\sigma^z_1,$$

and $$H_R = -\frac{1}{2}\sum_{n<0}\left(\sigma^x_i\sigma^x_{i-1} + \sigma^y_i\sigma^y_{i-1} + \Delta\sigma^z_i\sigma^z_{i-1}\right) - h\sigma^z_0.$$

The Spaces

- The operators act on $\mathcal{F}^{(i)} = \mathcal{H}_L^{(i)} \otimes \mathcal{H}_R^{(i)}$ where

$$\begin{aligned} \mathcal{H}_L^{(i)} &= \mathrm{Span}\{\cdots \otimes v_{\varepsilon(2)} \otimes v_{\varepsilon(1)} | \varepsilon(n) = (-1)^{n+i}, n \gg 0\}, \\ \mathcal{H}_R^{(i)} &= \mathrm{Span}\{v_{\varepsilon(0)} \otimes v_{\varepsilon(-1)} \otimes \cdots | \varepsilon(n) = (-1)^{n+i}, n \ll 0\}. \end{aligned}$$

$$T(\zeta) : \mathcal{F}^{(i)} \to \mathcal{F}^{(1-i)}, \quad T'(\zeta) : \mathcal{F}^{(i)} \to \mathcal{F}^{(i)}.$$

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$$T(\zeta) : \mathcal{F}^{(i)} \to \mathcal{F}^{(1-i)}, \quad T'(\zeta) : \mathcal{F}^{(i)} \to \mathcal{F}^{(i)}.$$

- Can identify $\mathcal{H}_R^{(i)}$ with $\mathcal{H}_L^{*(i)}$ and so

$$\mathcal{F}^{(i)} = \mathcal{H}_L^{(i)} \otimes \mathcal{H}_R^{(i)} \simeq \mathcal{H}_L^{(i)} \otimes \mathcal{H}_L^{*(i)} \simeq \text{End}(\mathcal{H}_L^{(i)}).$$

Object of Interest

- Interested in diagonalising $T(z)$ and $T(\zeta)'$, and computing ${}_{(i)}\langle \text{vac}| E^{\varepsilon_m}_{\varepsilon'_m} \cdots E^{\varepsilon_2}_{\varepsilon'_2} E^{\varepsilon_1}_{\varepsilon'_1} |\text{vac}\rangle'_{(i)}$, where $E^{\varepsilon}_{\varepsilon'}(v_a) = \delta_{a,\varepsilon} v_{\varepsilon'}$.

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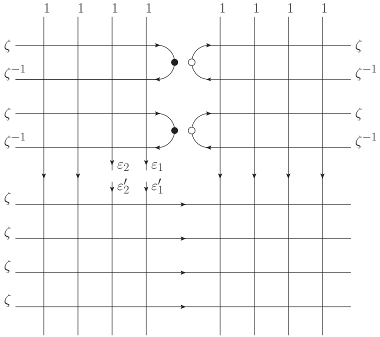
- Equal to



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- This geometry is very natural in VO approach
- Potential physical significance:
  - In description of *local quantum quench*:
    split 1D system at $t < 0$ brought together at $t = 0$
  - Time evolution of entanglement entropy between two halves studied using CFT - Cardy & Calabrese (2007)
  - $|\langle \text{vac}|\text{vac}\rangle'|^2$ called *fidelity*.

    -log(fidelity) suggested as alternative measure of quantum entanglement

    Studied using CFT by Dubail & Stéphan (2011).
  - Should be possible to relate our corr. fns. to CFT/lattice results in wedge of angle α, as $\alpha \to 2\pi$ - Cardy (1983), Barber et al (1984).

The Vertex Operator Approach - Space & Operators

- Identify $\mathcal{H}_L^{(i)}$ with $V(\Lambda_i)$ as $U_q(\widehat{\mathfrak{sl}}_2)$ modules, and hence

$$\mathcal{F}^{(i)} \simeq V(\Lambda_i) \otimes V(\Lambda_i)^* \simeq \mathrm{End}(V(\Lambda_i)).$$

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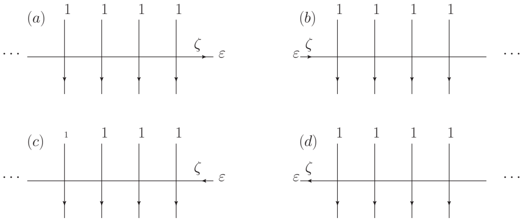
- Define VO:

$$\Phi(\zeta) : V(\Lambda_i) \xrightarrow{\sim} V(\Lambda_{1-i}) \otimes V_\zeta, \quad \Phi^*(\zeta) : V(\Lambda_i) \otimes V_\zeta \xrightarrow{\sim} V(\Lambda_{1-i}),$$
$$V_\zeta = V \otimes \mathbb{C}[[\zeta, \zeta^{-1}]].$$

- Define cpts $\Phi_\pm(\zeta), \Phi^*_\pm(\zeta) : V(\Lambda_i) \to V(\Lambda_{1-i})$

  and transposes $\Phi_\pm(\zeta)^t, \Phi^*_\pm(\zeta)^t : V(\Lambda_i)^* \to V(\Lambda_{1-i})^*$.
- Useful to note $\Phi^*_\varepsilon(\zeta) = \Phi_{-\varepsilon}(-q^{-1}\zeta^{-1})$

- Then identify



with

$$(a) \quad g^{\frac{1}{2}}\,\Phi_{\varepsilon}(\zeta), \qquad (b) \quad g^{\frac{1}{2}}\,\Phi_{-\varepsilon}(\zeta)^{t},$$
$$(c) \quad g^{\frac{1}{2}}\,\Phi^{*}_{\varepsilon}(\zeta), \qquad (d) \quad g^{\frac{1}{2}}\,\Phi^{*}_{-\varepsilon}(\zeta)^{t}.$$

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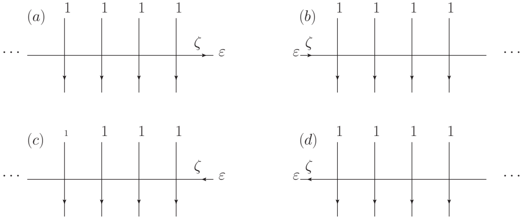

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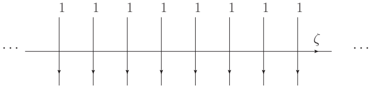
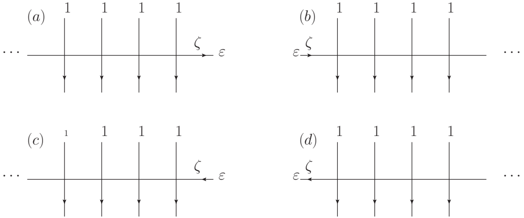
Hence Bulk $T(\zeta) =$



given by $T(\zeta) = g\sum_{\varepsilon}\Phi_{\varepsilon}(\zeta)\otimes\Phi_{-\varepsilon}(\zeta)^{t}$.



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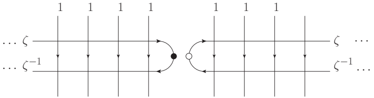
Frac. $T'(\zeta) =$  $= T_L(\zeta) \otimes T_R(\zeta)$, with

$$T_L(\zeta) = g \sum_{\varepsilon,\varepsilon'} \Phi^{*}_{\varepsilon'}(\zeta^{-1}) K_{\bullet\,\varepsilon'}^{\;\varepsilon}(\zeta) \Phi_{\varepsilon}(\zeta), \quad T_R(\zeta) = g \sum_{\varepsilon,\varepsilon'} \Phi^{*}_{-\varepsilon'}(\zeta^{-1})^{t} K_{\circ\,\varepsilon}^{\;\varepsilon'}(\zeta) \Phi_{-\varepsilon}(\zeta)^{t}$$
$$= T_L(-q^{-1}\zeta^{-1})^{t}$$

Eigenstates

- $T(\zeta)|\text{vac}\rangle_{(i)} = |\text{vac}\rangle_{(1-i)}$ eigenstate identified in [JM] as

$$|\text{vac}\rangle_{(i)} = \frac{1}{\chi^{\frac{1}{2}}}(-q)^D \in \text{End}(V(\Lambda_i)), \quad \text{with} \quad \chi = \text{Tr}_{V(\Lambda_i)}((-q)^{2D}).$$

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- $T_L(\zeta)|i\rangle_B = \Lambda(\zeta)^{(i)}|i\rangle_B$, ${}_B\langle i|T_L(\zeta) = \Lambda(\zeta)^{(i)}{}_B\langle i|$ vacuum eigenstates indentified in [JKKKM] as

$$|i\rangle_B = e^{F_i}|\Lambda_i\rangle, \quad {}_B\langle i| = \langle\Lambda_i|e^{G_i},$$

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- Hence, $T'(\zeta) = T_L(\zeta) \otimes T_L(-q^{-1}\zeta^{-1})^t$ eigenstate is just

$$|\text{vac}\rangle'_{(i)} = \frac{1}{{}_B\langle i|i\rangle_B}|i\rangle_B \otimes {}_B\langle i| \in V(\Lambda_i) \otimes V(\Lambda_i)^*, \quad \text{or}$$

$$|\text{vac}\rangle'_{(i)} = \frac{1}{{}_B\langle i|i\rangle_B}|i\rangle_{B\,B}\langle i| \in \text{End}(V(\Lambda_i)).$$

Correlation Functions

- Let us define (N even)

$$P^{(i)}(\zeta_1,\zeta_2,\cdots,\zeta_N) := \frac{1}{{}_{(i)}\langle \mathrm{vac}|\mathrm{vac}\rangle'_{(i)}}\,{}_{(i)}\langle \mathrm{vac}|\Phi(\zeta_1)\Phi(\zeta_2)\cdots\Phi(\zeta_N)\otimes \mathbb{I}|\mathrm{vac}\rangle'_{(i)},$$

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- Using $|\mathrm{vac}\rangle_{(i)} = \frac{1}{\chi^{\frac{1}{2}}}(-q)^D$, $|\mathrm{vac}\rangle'_{(i)} = \frac{1}{{}_B\langle i|i\rangle_B}|i\rangle_{B\,B}\langle i|$ gives

$$P^{(i)}(\zeta_1,\zeta_2,\cdots,\zeta_N) = \frac{1}{{}_B\langle i|(-q)^{D^{(i)}}|i\rangle_B}\,{}_B\langle i|(-q)^{D^{(i)}}\Phi(\zeta_1)\Phi(\zeta_2)\cdots\Phi(\zeta_N)|i\rangle_B.$$

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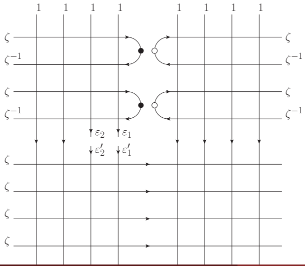
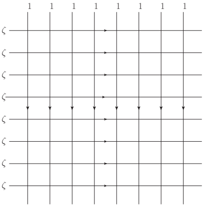
- 

$$= \quad g^m P^{(i)}(\zeta_1,\zeta_2,\cdots,\zeta_{2m})_{-\varepsilon'_1,\cdots,-\varepsilon'_m,\varepsilon_m,\cdots,\varepsilon_1},$$

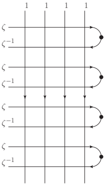
with $\zeta_1 = \zeta_2 = \ldots = \zeta_m = -q^{-1}$,

and $\zeta_{m+1} = \zeta_{m+2} = \ldots = \zeta_{2m} = 1.$

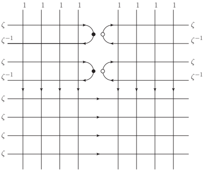
Alternative CTM Approach - 3 partition functions



$$Z_{bulk} = \mathrm{Tr}_{\mathcal{H}_L^{(i)}}(A^{(i)}_{NE}(\zeta)A^{(i)}_{SE}(\zeta)A^{(i)}_{SW}(\zeta)A^{(i)}_{NW}(\zeta))$$



$$Z_{boundary} = {}^{(i)}_{\bullet}\langle B;\zeta|A^{(i)}_{SW}(\zeta,1)A^{(i)}_{NW}(\zeta,1)|B;\zeta\rangle^{(i)}_{\bullet}$$



$$Z_{fracture} = {}^{(i)}_{\circ}\langle B;\zeta|A^{(i)}_{NE}(\zeta,1)A^{(i)}_{SE}(\zeta)A^{(i)}_{SW}(\zeta)A^{(i)}_{NW}(\zeta,1)|B;\zeta\rangle^{(i}_{\bullet}$$

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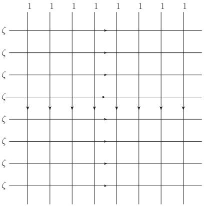

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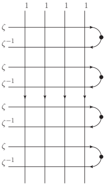
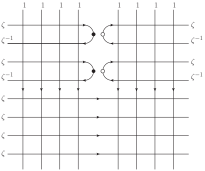


$$Z_{fracture} = {}^{(i)}_{\circ}\langle B;\zeta|A_{NE}^{(i)}(\zeta,1)A_{SE}^{(i)}(\zeta)A_{SW}^{(i)}(\zeta)A_{NW}^{(i)}(\zeta,1)|B;\zeta\rangle^{(i)}_{\bullet}$$

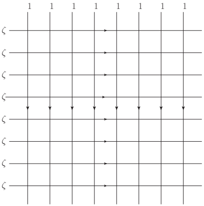


1) Use $A_{SW}(\zeta) \sim \zeta^{-D}$

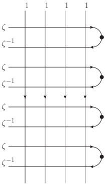
2) Use xing symmetry to relate different CTMs

3) Let $|i\rangle_B \sim A_{NW}^{(i)}(\zeta,1)|B;\zeta\rangle^{(i)}_{\bullet}$, ${}_B\langle i| \sim {}^{(i)}_{\bullet}\langle B;\zeta|A_{SW}^{(i)}(\zeta,1)$, to get

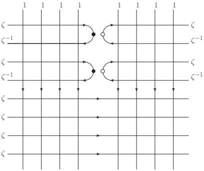
Alternative CTM Approach - 3 partition functions



$$Z_{bulk} = \mathrm{Tr}_{\mathcal{H}_L^{(i)}}(q^{2D})$$



$$Z_{boundary} = {}_B\langle i|i\rangle_B$$



$$Z_{fracture} = {}_B\langle i|(-q)^D|i\rangle_B$$

The Boundary qKZ Equation

- Interested in

$$G^{(i)}(\zeta_1, \zeta_2, \cdots, \zeta_N) = \frac{1}{{}_B\langle i|i\rangle_B} {}_B\langle i|\Phi_{\varepsilon_1}(\zeta_1)\Phi_{\varepsilon_2}(\zeta_2)\cdots\Phi_{\varepsilon_N}(\zeta_N)|i\rangle_B,$$

$$P^{(i)}(\zeta_1, \zeta_2, \cdots, \zeta_N) = \frac{1}{{}_B\langle i|(-q)^D|i\rangle_B} {}_B\langle i|(-q)^D\Phi_{\varepsilon_1}(\zeta_1)\Phi_{\varepsilon_2}(\zeta_2)\cdots\Phi_{\varepsilon_N}(\zeta_N)|i\rangle_B$$

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- Following hold:

$$\begin{aligned}
K(\zeta)\Phi(\zeta)|i\rangle_B &= \Lambda^{(i)}(\zeta;r)\phi(\zeta^{-1})|i\rangle_B\\
\hat{K}(-q^{-1}\zeta)\ {}_B\langle i|\Phi(\zeta^{-1}) &= \Lambda^{(i)}(-q^{-1}\zeta;r){}_B\langle i|\Phi(q^{-2}\zeta)\\
\hat{K}(q^{-2}\zeta)\ {}_B\langle i|(-q)^D\Phi(\zeta^{-1}) &= \Lambda^{(i)}(q^{-2}\zeta;r){}_B\langle i|(-q)^D\Phi(q^{-4}\zeta)\\
PR(\zeta_1/\zeta_2)\Phi(z_1)\Phi(\zeta_2) &= \Phi(\zeta_2)\Phi(\zeta_1)
\end{aligned}$$

where $\hat{K}_{\varepsilon}^{\varepsilon'}(\zeta)=K_{-\varepsilon'}^{-\varepsilon}(\zeta)$.

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where $\hat{K}_{\varepsilon}^{\varepsilon'}(\zeta)=K_{-\varepsilon'}^{-\varepsilon}(\zeta)$.

- Insert into correlation fns to give boundary qKZ:

- $$G^{(0)}(\zeta_1,\cdots,\zeta_{j-1},q^{-2}\zeta_j,\zeta_{j+1},\cdots,\zeta_N) =$$
$$R_{j,j-1}(\zeta_j/q^2\zeta_{j-1})\cdots R_{j,1}(\zeta_j/q^2\zeta_1)\hat{K}_j(-q^{-1}\zeta_j)$$
$$\times \quad R_{1j}(\zeta_1\zeta_j)\cdots R_{j-1,j}(\zeta_{j-1}\zeta_j)R_{j+1,j}(\zeta_{j+1}\zeta_j)\cdots R_{nj}(\zeta_n\zeta_j)$$
$$\times \quad K_j(\zeta_j)R_{j,N}(\zeta_j/\zeta_N)\cdots R_{j,j+1}(\zeta_j/\zeta_{j+1})G^{(0)}(\zeta_1,\zeta_2.\cdots,\zeta_N),$$

and
$$P^{(0)}(\zeta_1,\cdots,\zeta_{j-1},q^{-4}\zeta_j,\zeta_{j+1},\cdots,\zeta_N) =$$
$$R_{j,j-1}(\zeta_j/q^4\zeta_{j-1})\cdots R_{j,1}(\zeta_j/q^4\zeta_1)\hat{K}_j(-q^{-2}\zeta_j)$$
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- If $\Psi(\zeta_1,\cdots,\zeta_N)$ related to $\Psi(\zeta_1,\cdots,r^{\frac{1}{2}}s^{\frac{1}{2}}/\zeta_N)$ and $\Psi(r^{\frac{1}{2}}/\zeta_1,\zeta_2,\cdots,\zeta_N)$, then qKZ of type (r,s), and $s=q^{2(2+\ell)}$ defines level ℓ [PdiF:math-ph/0509011].

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- Hence $(r,s)=(q^{-4},q^4)$ with level $=0$ for boundary
  $(r,s)=(q^{-8},q^8)$ with level $=2$ for fracture

Integral Expression

- Use free-field realisation to give integral expression for

$$P^{(i)}(\zeta_1, \zeta_2, \cdots, \zeta_N) = \frac{1}{{}_B\langle i|(-q)^D|i\rangle_B} {}_B\langle i|(-q)^D \Phi_{\varepsilon_1}(\zeta_1)\Phi_{\varepsilon_2}(\zeta_2)\cdots\Phi_{\varepsilon_N}(\zeta_N)|i\rangle_B$$

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- Then specialise to give

$$\frac{1}{{}_{(i)}\langle \mathrm{vac}|\mathrm{vac}\rangle'_{(i)}}\,{}_{(i)}\langle \mathrm{vac}|E^{\varepsilon_m}_{\varepsilon'_m}\cdots E^{\varepsilon_2}_{\varepsilon'_2}E^{\varepsilon_1}_{\varepsilon'_1}|\mathrm{vac}\rangle'_{(i)} =$$

$$g^m P^{(i)}(\zeta_1,\zeta_2,\cdots,\zeta_{2m})_{-\varepsilon'_1,\cdots,-\varepsilon'_m,\varepsilon_m,\cdots,\varepsilon_1},$$

with the choice

$$\zeta_1=\zeta_2=\ldots=\zeta_m=-q^{-1},\quad \zeta_{m+1}=\zeta_{m+2}=\ldots=\zeta_{2m}=1.$$

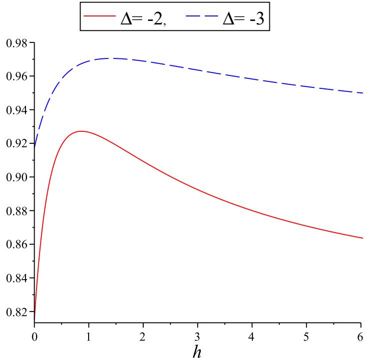
The overlap ${}_{(i)}\langle \mathrm{vac}|\mathrm{vac}\rangle'_{(i)}$

We find $\left[\text{with } (a; b_1, \cdots, b_N)_\infty = \prod_{j_1,\cdots,j_N=0}^{\infty}(1 - a b_1^{j_1} \cdots b_N^{j_N})\right]$

$$ {}_{(0)}\langle \mathrm{vac}|\mathrm{vac}\rangle'_{(0)} $$

$$ = (q^2;q^4)_\infty^{\frac{1}{2}} \frac{(r^2q^{10};q^8,q^8)_\infty^2}{(r^2q^4;q^8,q^8)_\infty (r^2q^{12};q^8,q^8)_\infty} \frac{(r^2q^2;q^4,q^8)_\infty}{(r^2q^4;q^4,q^8)_\infty} \frac{(q^6;q^8,q^8)_\infty}{(q^{10};q^8,q^8)_\infty} $$

$\left|{}_{(0)}\langle \mathrm{vac}|\mathrm{vac}\rangle'_{(0)}\right|^2$ vs h



Fracture Magnetization

- We have

$$M^{(i)}(r) := \frac{{}_{(i)}\langle \text{vac}|\sigma_1^z|\text{vac}\rangle'_{(i)}}{{}_{(i)}\langle \text{vac}|\text{vac}\rangle'_{(i)}} = g\left(P^{(i)}(-q^{-1},1)_{-+} - P^{(i)}(-q^{-1},1)_{+-}\right).$$

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Defining $z := \zeta^2$, get $gP^{(i)}_{+-}(-q^{-1}\zeta,\zeta) =$

$$-(q^2z)^i z(1-q^2)^2 \oint_{C^{(i)}_{+-}} \frac{dw}{2\pi\sqrt{-1}} \frac{w^{1-i}}{(w-z)(w-q^2z)(w-q^4z)} I'^{(i)}$$

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$$\begin{aligned} I'^{(i)} \quad = \quad & F^{(i)}\,(q^8 z^2;q^8)_\infty (q^4/z^2;q^8)_\infty (q^8;q^8)_\infty (q^{10};q^8)^2_\infty \Theta_{q^8}(q^2 w^2) \\ & \times \frac{1}{(q^6 zw;q^8)_\infty (q^4/(zw);q^8)_\infty (q^{12} z/w;q^8)_\infty (q^6 w/z;q^8)_\infty} \\ & \times \frac{1}{(q^4 zw;q^8)_\infty (q^6/(zw);q^8)_\infty (q^{10} z/w;q^8)_\infty (q^8 w/z;q^8)_\infty}, \end{aligned}$$

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and

$$F^{(0)} = \frac{(q^2rz;q^8)_\infty(q^4r/z;q^8)_\infty}{(q^8rz;q^8)_\infty(q^2r/z;q^8)_\infty}\frac{(q^6rw;q^8)_\infty(q^4r/w;q^8)_\infty}{(rw;q^8)_\infty(q^6r/w;q^8)_\infty},$$

$$F^{(1)} = \frac{(1/(rz);q^8)_\infty(q^6z/r;q^8)_\infty}{(q^6/(rz);q^8)_\infty(q^4z/r;q^8)_\infty}\frac{(q^2w/r;q^8)_\infty(q^8/(rw);q^8)_\infty}{(q^4w/r;q^8)_\infty(q^2/(rw);q^8)_\infty}.$$

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- Conjecture (correct to at least $O(q^{96})$):

$$M^{(0)}(r) = -1 - 2(1-r)\sum_{n=1}^{\infty} \frac{(-q^2)^n}{(1-rq^{4n})}$$

$$\text{c.f.} \quad M^{(0)}_{bound}(r) = -1 - 2(1-r)^2\sum_{n=1}^{\infty} \frac{(-q^2)^n}{(1-rq^{2n})^2}$$

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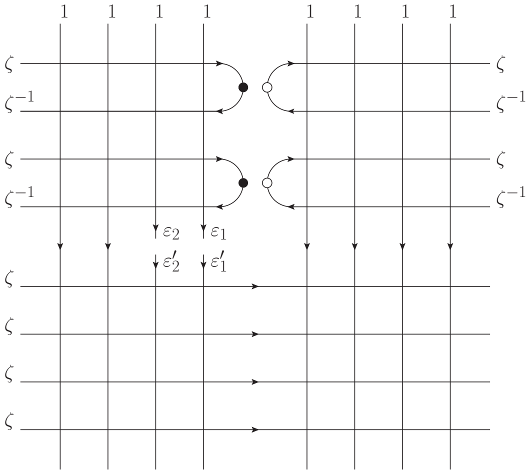
- Special points:

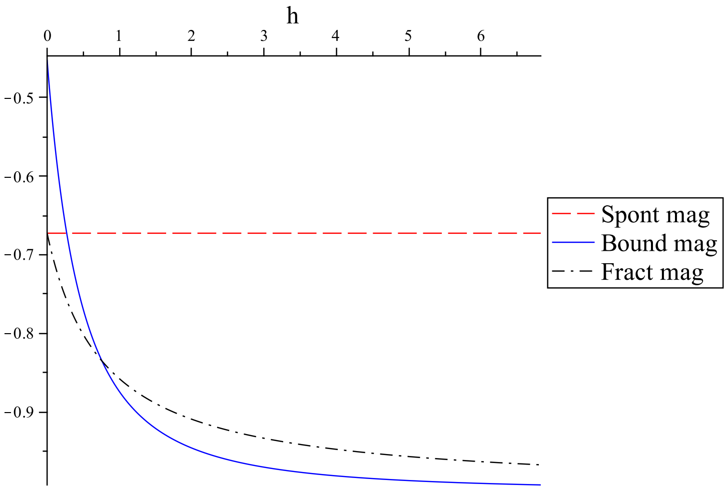
$$M^{(0)}(r=-1) = -\frac{(q^2;q^2)^2_\infty}{(-q^2;q^2)^2_\infty}, \quad h=0$$

$$M^{(0)}(r=0) = M^{(0)}_{bound}(r=0) = -\frac{1-q^2}{1+q^2}, \quad h=h_{inv}$$

$$M^{(0)}(r=1) = M^{(1)}_{bound}(r=1) = -1, \quad h=\infty.$$

Recall picture:




h
0
1
2
3
4
5
6
−0.5
−0.6
−0.7
−0.8
−0.9
Spont mag
Bound mag
Fract mag

Summary & Discussion

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- This is a natural geometry to consider
- Differences to pure boundary case explained by extra $(-q)^D$ in correlation functions
  - qKZ of different level
  - integral formula more complicated
- Integral formula are efficient way of giving q-expansions
- Generalisable to RSOS, sl_n, elliptic case ...
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Thank you