Symmetry, Curves and Monopoles

H.W. Braden

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Curve results with T.P. Northover.
Monopole Results in collaboration with V.Z. Enolski, A.D'Avanzo.

Overview

Equations	$\overset{Zero\ \ Curvature/Lax}{\longrightarrow}$	Spectral Curve $\mathcal{C} \subset \mathcal{S}$
$\uparrow$		$\downarrow$
Reconstruction	$\longleftarrow$	Baker-Akhiezer Function

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- BPS Monopoles
- Sigma Model reductions in AdS/CFT
- KP, KdV solitons
- Harmonic Maps
- SW Theory/Integrable Systems

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$\theta\left(t\mathbf{U} + \mathbf{C}|\tau\right)$

- Reduction of $F = *F$

$$L = -\frac{1}{2}\mathrm{Tr}\, F_{ij}F^{ij} + \mathrm{Tr}\, D_i\Phi\, D^i\Phi.$$

- $B_i = \frac{1}{2}\sum_{j,k=1}^{3} \epsilon_{ijk}F^{jk} = D_i\Phi$
- A *monopole* of charge n

$$\sqrt{-\frac{1}{2}\mathrm{Tr}\,\Phi(r)^2}\Bigg|_{r\to\infty} \sim 1 - \frac{n}{2r} + O(r^{-2}), \quad r = \sqrt{x_1^2 + x_2^2 + x_3^2}$$

- Monopoles $\leftrightarrow$ Nahm Data $\leftrightarrow$ Hitchin Data

Three $n \times n$ matrices $T_i(s)$ with $s \in [0,2]$ satisfying the following:

- N1 Nahm's equation $$\frac{dT_i}{ds} = \frac{1}{2} \sum_{j,k=1}^{3} \epsilon_{ijk} [T_j, T_k].$$
- N2 $T_i(s)$ is regular for $s \in (0,2)$ and has simple poles at $s = 0, 2$. Residues form $su(2)$ irreducible n-dimensional representation.
- N3 $T_i(s) = -T_i^{\dagger}(s), \qquad T_i(s) = T_i^t(2-s).$

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$$A(\zeta) = T_1 + iT_2 - 2iT_3\zeta + (T_1 - iT_2)\zeta^2$$
$$M(\zeta) = -iT_3 + (T_1 - iT_2)\zeta$$

Nahm's eqn. $$\frac{dT_i}{ds} = \frac{1}{2}\sum_{j,k=1}^{3} \epsilon_{ijk}[T_j, T_k] \iff [\frac{d}{ds} + M, A] = 0.$$

Spectral Curves

$\mathcal{C} \subset \mathcal{S}$

- $[\frac{d}{ds} + M(\zeta), A(\zeta)] = 0, \quad \mathcal{C}: \; 0 = \det(\eta 1_n + A(\zeta)) := P(\eta, \zeta)$

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- Symmetry: $\mathcal{C} \subset \mathbb{P}^{a,b,c} \quad [X, Y, Z] \sim [\lambda^a X, \lambda^b Y, \lambda^c Z], \; \lambda \in \mathbb{C}^*$

Spectral Curves: data

- Homology basis $\{\gamma_i\}_{i=1}^{2g} = \{\mathfrak{a}_i, \mathfrak{b}_i\}_{i=1}^{g}$
  - algorithm for branched covers of $\mathbb{P}^1$ (Tretkoff & Tretkoff)
  - poor if curve has symmetries
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  - reality constrains the form of the period matrix.
  - there may be between 0 and $g+1$ ovals of fixed points of the antiholomorphic involution.
  - Imposing reality can be one of the hardest steps.

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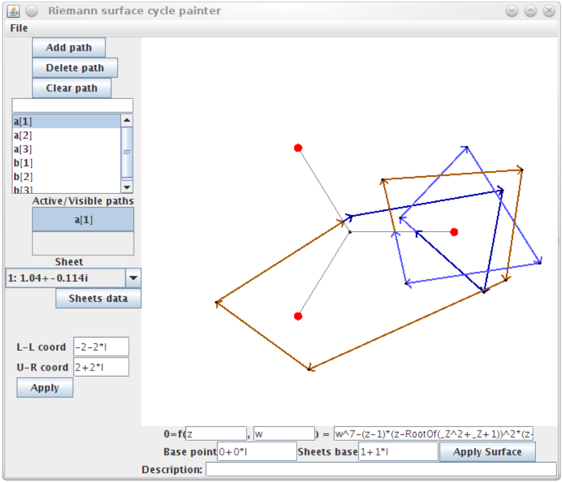

Riemann surface cycle painter
File
Add path
Delete path
Clear path
a[1]
a[2]
a[3]
b[1]
b[2]
b[3]
Active/Visible paths
a[1]
Sheet
1: 1.04+-0.114i
Sheets data
L-L coord -2-2*I
U-R coord 2+2*I
Apply
0=f(z , w) = w^7-(z-1)*(z-RootOf(_Z^2+_Z+1))^2*(z-
Base point 0+0*I
Sheets base 1+1*I
Apply Surface
Description:

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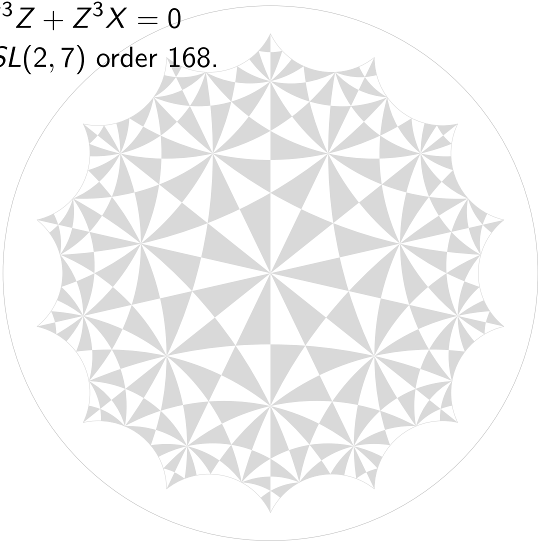
Curves with lots of symmetries: evaluate τ via character theory

- How can one specify homology cycles?
- How to determine M, $\sigma_*(\gamma) = M.\gamma$? `extcurves`
- How to determine a good basis $\{\gamma_i\}$?

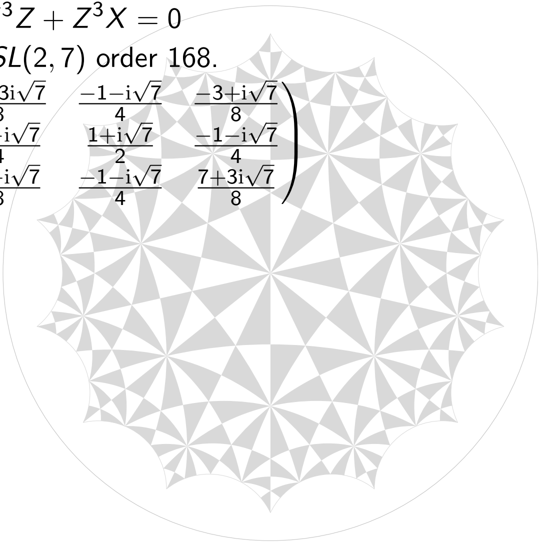
- $\mathcal{C}$: $X^3Y + Y^3Z + Z^3X = 0$
- $\mathrm{Aut}(\mathcal{C}) = PSL(2,7)$ order 168.



Calculation

Example: Klein's Curve and Problems

- $\mathcal{C}$: $X^3Y + Y^3Z + Z^3X = 0$
- $\mathrm{Aut}(\mathcal{C}) = PSL(2,7)$ order 168.
- $\tau_{RL} = \begin{pmatrix} \frac{-1+3\mathrm{i}\sqrt{7}}{8} & \frac{-1-\mathrm{i}\sqrt{7}}{4} & \frac{-3+\mathrm{i}\sqrt{7}}{8} \\ \frac{-1-\mathrm{i}\sqrt{7}}{4} & \frac{1+\mathrm{i}\sqrt{7}}{2} & \frac{-1-\mathrm{i}\sqrt{7}}{4} \\ \frac{-3+\mathrm{i}\sqrt{7}}{8} & \frac{-1-\mathrm{i}\sqrt{7}}{4} & \frac{7+3\mathrm{i}\sqrt{7}}{8} \end{pmatrix}$



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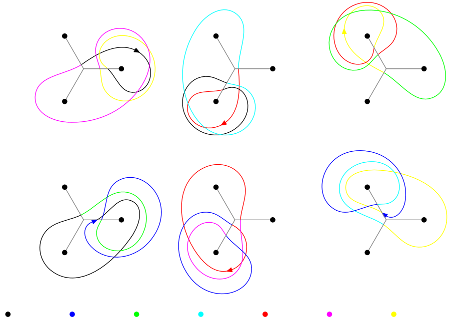

Figure: Homology basis in (z, w) coordinates

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- $\tau = \dfrac{1}{2}\begin{pmatrix} e & 1 & 1 \\ 1 & e & 1 \\ 1 & 1 & e \end{pmatrix}, \quad e = \frac{-1+\mathrm{i}\sqrt{7}}{2}$

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- This depends on finding a good adapted basis simplifying the action of $\mathrm{Aut}(\mathcal{C})$ on $H_1(\mathcal{C},\mathbb{Z})$
- Symplectic Equivalence of Period Matrices τ, τ'

  $M = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in Sp(2g,\mathbb{Z}) \Leftrightarrow M^T J M = J$

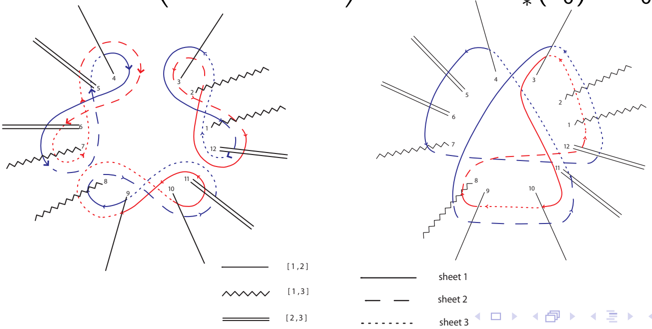
  $\begin{pmatrix} \tau' & -1 \end{pmatrix} M \begin{pmatrix} 1 \\ \tau \end{pmatrix} = 0$

$$\hat{\mathcal{C}}: \qquad w^3 + \alpha w z^2 + \beta z^6 + \gamma z^3 - \beta = 0$$

$$\mathrm{C}_3 : (z, w) \mapsto (\rho z, \rho w), \;\; \rho = \exp(2\pi \mathrm{i}/3)$$

$$\tau_{\hat{\mathcal{C}}\mathrm{monopole}} = \begin{pmatrix} a & b & b & b \\ b & c & d & d \\ b & d & c & d \\ b & d & d & c \end{pmatrix}$$

$$\sigma_*^k(\mathfrak{a}_i) = \mathfrak{a}_{i+k}$$
$$\sigma_*^k(\mathfrak{b}_i) = \mathfrak{b}_{i+k}$$
$$\sigma_*^k(\mathfrak{a}_0) = \mathfrak{a}_0$$
$$\sigma_*^k(\mathfrak{b}_0) \sim \mathfrak{b}_0$$



Calculation

The spectral curve of genus 2

$$\mathcal{C} = \hat{\mathcal{C}}/C_3 : \qquad y^2 = (x^3 + \alpha x + \gamma)^2 + 4\beta^2$$

$$\tau = \begin{pmatrix} \frac{a}{3} & b \\ b & c + 2d \end{pmatrix}$$

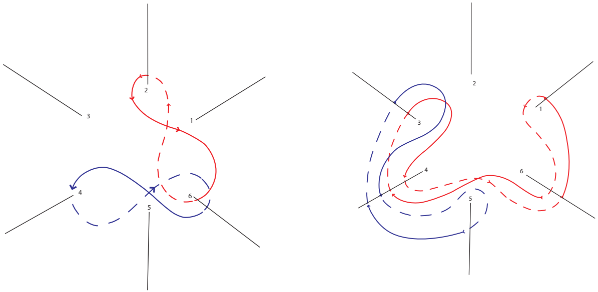


Figure: Projection of the previous basis

H1 $\mathcal{C} \subset T\mathbb{P}^1$ Reality conditions $a_r(\zeta) = (-1)^r \zeta^{2r} \overline{a_r(-\frac{1}{\bar{\zeta}})}$

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$\Longleftrightarrow$ Ercolani-Sinha Constraints:
$$2\mathbf{U} \in \Lambda \Longleftrightarrow \mathbf{U} = \frac{1}{2\pi\imath}\left(\oint_{\mathfrak{b}_1}\gamma_\infty, \ldots, \oint_{\mathfrak{b}_g}\gamma_\infty\right)^T = \frac{1}{2}\mathbf{n} + \frac{1}{2}\tau\mathbf{m}$$

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H3 $H^0(\mathcal{C}, \mathcal{L}^s(n-2)) = 0$ for $s \in (0,2) \Longleftrightarrow \theta(s\mathbf{U} + \mathbf{C} \,|\, \tau) \neq 0$

$$\mathbf{C} = K_Q + \phi_Q\left((n-2)\sum_{k=1}^{n} \infty_k\right)$$

Second transcendental constraint.

Cyclically Symmetric Monopoles

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$$\begin{pmatrix} p & q \\ -\bar{q} & \bar{p} \end{pmatrix} \in PSU(2), \qquad |p|^2 + |q|^2 = 1,$$

$$\zeta \to \frac{\bar{p}\,\zeta - \bar{q}}{q\,\zeta + p}, \qquad \eta \to \frac{\eta}{(q\,\zeta + p)^2}$$

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  $\mathtt{C}_n$ symmetric (centred) charge-n monopole curve of form
  $$\hat{\mathcal{C}}: \quad \eta^n + a_2\eta^{n-2}\zeta^2 + \ldots + a_n\zeta^n + \beta\zeta^{2n} + (-1)^n\beta = 0, \quad a_i, \beta \in \mathbf{R}$$
- $\hat{\mathcal{C}}$ a $n:1$ unbranched cover Affine Toda Spectral Curve
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Theorem

Any cyclically symmetric monopole is gauge equivalent to Nahm data given by Sutcliffe's ansatz, and so obtained from the affine Toda equations.

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- Fay-Accola

$$\theta[\mathbf{C}](\pi^* z;\, \tau_{\mathrm{monopole}}) = c \prod_{i=1}^{n} \theta[\mathbf{e}_i](z;\, \tau_{\mathrm{Toda}})$$

"*θ-functions are still far from being a spectator sport.*"(L.V. Ahlfors)

C_3 Cyclically Symmetric Monopoles

- $\mathfrak{c} := \pi(\mathfrak{es})$
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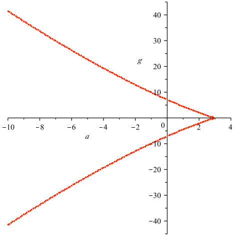

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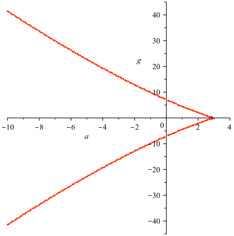

- With $a = \alpha/\beta^{2/3}$, $g = \gamma/\beta$ and β defined by

  $$6\beta^{1/3} = \oint_{\mathfrak{c}} \frac{X\mathrm{d}X}{Y}$$

  we may recover the monopole spectral curve.