Instantons and Killing spinors

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EMPG

Outline

1. Introduction and motivation
2. Instantons on real Killing spinor manifolds
3. Instantons on the cone
4. Heterotic supergravity
5. Conclusions

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Instanton equations

Definition

A gauge field A is called an instanton if its field strength F satisfies

$$F \cdot \epsilon = 0,$$

for some spinor ϵ.

This is equivalent to

$$\frac{1}{2} Q_{\mu\nu\kappa\lambda} F^{\kappa\lambda} = -F_{\mu\nu}$$

where $Q_{\mu\nu\kappa\lambda} = \langle \epsilon | \gamma_{\mu\nu\kappa\lambda} | \epsilon \rangle$ (cf. Corrigan, Fairlie, Devchand, Nuyts, 1983).

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Example (4 dimensions)

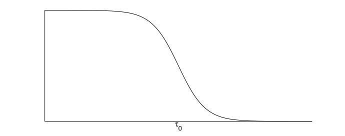
If ϵ is a Weyl spinor, $F \cdot \epsilon = 0$ is equivalent to

$$\frac{1}{2}\varepsilon_{\mu\nu\kappa\lambda}F^{\kappa\lambda} = -F_{\mu\nu}.$$

Instantons solve the Yang-Mills equation $D_\mu F^{\mu\nu} = 0$.

The BPST instanton is a kink:

$$\begin{aligned} A &= (1-\psi)e^a I_a \\ \psi &= \left(1 + e^{2(\tau-\tau_0)}\right)^{-1} \end{aligned}$$



Here $\tau = \ln r$ is a radial coordinate, I_a are generators for SU(2), e^a are left-invariant 1-forms on S^3.

Applications

- Implies the Yang-Mills equation, if ϵ is *parallel* (e.g. on $\mathbb{R}^n$).
- BPS states in super-Yang-Mills, heterotic supergravity.
- Invariants of n-manifolds, $n > 4$.
- Reductions appear in geometric Langlands, complex Chern-Simons, self-dual strings.

Some solutions

- The Levi-Civita connection on any manifold with parallel spinor is an instanton (e.g. Calabi-Yau manifolds, hyper-Kähler manifolds).
- Model solutions on $\mathbb{R}^n$ (ϵ fixed by $G \subset SO(n)$):

n	G	instanton?	SUGRA?	name
7,8	G_2, Spin(7)	✓	✓	octonionic
$2m$ $m \geq 3$	SU(m)	✗	✗	complex
$4m$ $m \geq 1$	Sp(m)	✓	✗	quaternionic

Killing spinors and cones

Definition

$$\nabla_\mu \epsilon = \mathrm{i}\lambda\gamma_\mu \cdot \epsilon$$

- $\lambda = 0$: ϵ called a parallel spinor
- $\lambda \neq 0$: ϵ called a Killing spinor

Cone construction: $g_C = \mathrm{d}r^2 + r^2 g_M, r > 0.$

Killing spinors on M = parallel spinors on C.

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Classification

Theorem (Bär 1993)

Manifolds M with real Killing spinors are one of the following:

M	dim	cone	example
nearly parallel G_2	7	Spin(7)	S^7
nearly Kähler	6	Joyce	S^6
Sasaki-Einstein	$2m+1$, $m \geq 1$	Calabi-Yau	S^{2m+1}
3-Sasakian	$4m+3$, $m \geq 0$	hyperkähler	S^{4m+3}

(or are round spheres in other dimensions).

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Instantons

Is there a canonical instanton on the tangent bundle over a real Killing spinor manifold M?

Proposition

A connection on the tangent bundle with curvature tensor $R_{\mu\nu\kappa\lambda}$ is an instanton if

1. *it has a parallel spinor*
2. $R_{\mu\nu\kappa\lambda} = R_{\kappa\lambda\mu\nu}$

The Levi-Civita connection satisfies (2) but not (1).

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The characteristic connection

Definition (Agricola 2006)

A characteristic connection is a connection with reduced holonomy $G \subset \mathrm{SO}(n)$ and totally anti-symmetric torsion, such that the torsion 3-form is parallel.

Such connections satisfy (2) (Agricola 2006).

- NP, NK: $\exists!$ characteristic connection with holonomy G_2, SU(3) (Friedrich & Ivanov 2002). It is an instanton.
- SE: $\exists!$ characteristic connection with holonomy $\mathrm{U}(m)$ (FI'02). It is not an instanton.
- 3S: $\nexists$ a characteristic connection (Agricola & Friedrich '08).

The canonical connection

Definition

A canonical connection is a connection with reduced holonomy $G \subset SO(n)$ and totally anti-symmetric torsion with respect to some G-compatible metric, such that the torsion 3-form is parallel.

Such connections also satisfy (2).

- NP/NK: $\exists!$ canonical connection (= characteristic connection).
- SE: $\exists!$ characteristic connection with holonomy $SU(m)$.
  It is an instanton.
- 3S: $\exists$ characteristic connection with holonomy $Sp(m)$.
  It is an instanton.

Yang-Mills equation

Proposition

The instanton equation implies the Yang-Mills equation on a real Killing spinor manifold.

Proof.

Differentiate instanton equation $F \cdot \epsilon = 0$. □

Bogomolny-type argument:

$$-\int \mathrm{Tr}(F \wedge *F) \geq \int \mathrm{Tr}(F \wedge F) \wedge *Q$$

Lower bound is not topological!

Instanton equation $\Rightarrow$ EOM for lower bound + saturation of inequality.

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Instantons on the cone

Recall: the cone is $C = \mathbb{R}_{>0} \times M$, with metric

$$g_C = \mathrm{d}r^2 + r^2 g_M = e^{2\tau}(\mathrm{d}\tau^2 + g_M).$$

There are two obvious instantons on the cone:

- The Levi-Civita connection on C
- The canonical connection on M

Are there any more?

Nearly parallel G_2

Ansatz:

$$A = \text{canonical connection} + \psi(\tau) e^a I_a$$

- e^a are a local orthonormal frame for T^*M (vielbein).
- I_a are matrices constructed so that A has a parallel spinor.

Instanton equation is

$$\dot{\psi} = 2(\psi^2 - \psi).$$

Solution:

$$\psi(\tau) = \left(1 + e^{2(\tau-\tau_0)}\right)^{-1}.$$

Interpolates between Levi-Civita connnection (at $r = 0$) and canonical connection (at $r = \infty$).

Case $M = S^7$: get FNFN instanton on $\mathbb{R}^8$.

Nearly Kähler story similar, $M = S^6$ gives GN instanton on $\mathbb{R}^7$.

Consistent reductions

Remark: ansatz reduces PDE to ODE, *without* assuming M has symmetries! $\Rightarrow$ this is a consistent reduction (cf Gauntlett).

Our consistent reduction is based on the general holonomy principle: given a principle G-bundle,

representations of G	$\leftrightarrow$	vector bundles
trivial representations of G	$\leftrightarrow$	parallel sections

$e^a I_a$ is a parallel section corresponding to a trivial sub-representation of a representation of G_2.

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$(2m+1)$-dimensional Sasaki-Einstein I

Similar ansatz $\Rightarrow$ ODEs for 2 functions χ, ψ:

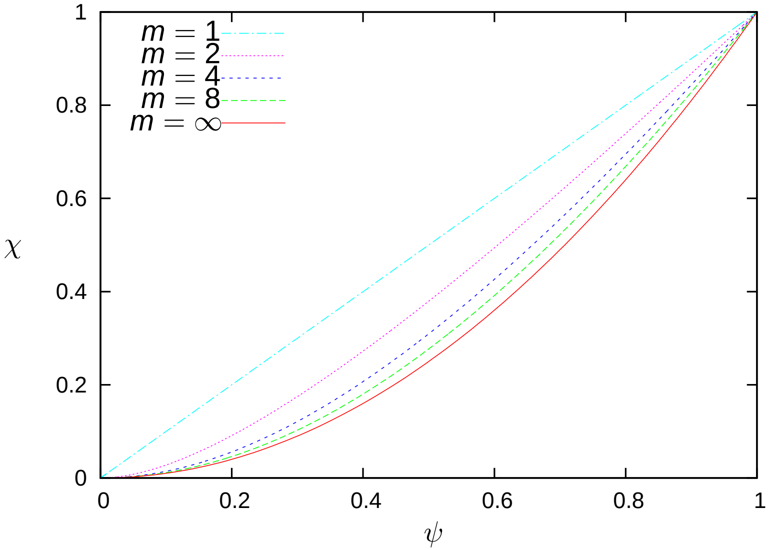
$$\begin{aligned} \dot{\chi} &= 2m(\psi^2 - \chi) \\ \dot{\psi} &= \frac{m+1}{m}\psi(\chi - 1). \end{aligned}$$

(cf Correia 2010). Numerical solutions only.

Interpolates between Levi-Civita connnection (at $r = 0$) and canonical connection (at $r = \infty$).

$$M = S^{2m+1}\text{: new instantons on } \mathbb{R}^{2m+2}.$$

$(2m+1)$-dimensional Sasaki-Einstein II



$(4m+3)$-dimensional 3-Sasakian

Obtain 3 equations for 2 functions:

$$\begin{aligned} 0 &= \chi - \psi^2 \\ \dot{\chi} &= 2\chi(\chi - 1) \\ \dot{\psi} &= \psi(\chi - 1). \end{aligned}$$

Nevertheless, there is an exact solution:

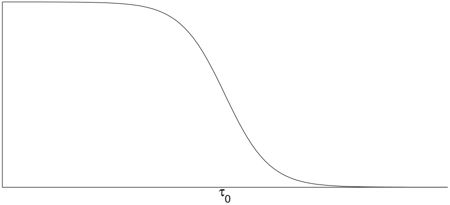
$$\begin{aligned} \chi(\tau) &= \left(1 + e^{2(\tau-\tau_0)}\right)^{-1} \\ \psi(\tau) &= \pm\left(1 + e^{2(\tau-\tau_0)}\right)^{-1/2}. \end{aligned}$$

Interpolates between Levi-Civita connnection (at $r = 0$) and canonical connection (at $r = \infty$).

$M = S^{4m+3}$: the CGK instantons on $\mathbb{R}^{4m+4}$.

Singular limit

Our instantons are domain walls, or kinks:



- Large size limit $\tau_0 \to \infty$: Levi-Civita connection on cone.
- Small size limit $\tau_0 \to -\infty$: canonical connection.
  Singular (even on $\mathbb{R}^n$).

The instantons provide models of singularity formation (cf Tian).

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Heterotic supergravity

The BPS equations of heterotic supergravity are

$$\begin{aligned} \nabla^{-}\epsilon &= 0 \\ (\mathrm{d}\phi - H)\cdot\epsilon &= 0 \\ F\cdot\epsilon &= 0 \\ \mathrm{d}H &= -\frac{\alpha'}{4}\mathrm{Tr}(F\wedge F - R^{+}\wedge R^{+}). \end{aligned}$$

The instantons on cones lift to solutions of these equations, at least to $O(\alpha')$ (generalises Harvey & Strominger 1990).

Explicit solutions in nearly parallel G_2, nearly Kähler, 3-Sasakian cases, numerical solutions in Sasaki-Einstein case.

3-Sasakian: solution exists despite having more equations than unkowns!

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Conclusions

- Instantons on real Killing spinor manifolds and their cones
- List of model solutions on $\mathbb{R}^n$ complete:

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- Demonstrate singularity-formation

- Resolutions of cones?
- Uniqueness of canonical connection?
- Multi-instantons?
- Twistors?