Symplectic structures on moduli spaces of framed sheaves on surfaces

Francesco Sala

May 16th, 2012

Gauge Theory

$$\Updownarrow$$

Algebraic Geometry

Motivations

Definition

An *$SU(r)$-instanton* on S^4 is a Hodge anti-selfdual (ASD) connection A on a principal $SU(r)$-bundle P on S^4.

Instantons are classified by their instanton number n.

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A *framed $SU(r)$-instanton* on S^4 is a pair (A, ϕ) where

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Theorem (Donaldson)

$$\{\text{framed } SU(r)\text{-instantons on } S^4 \text{ with instanton number } n\}/\sim$$

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$$\mathcal{M}_{lf}(r,n) := \{\text{vector bundles on } \mathbb{CP}^2$$
$$\text{of rank } r \text{ with second Chern class } n,$$
$$\text{that are trivial along a fixed line } l_\infty\}/\sim$$

We call these objects *framed vector bundles* on $\mathbb{CP}^2$.

$\mathcal{M}_{lf}(r,n)$ is an open subscheme of the moduli space $\mathcal{M}(r,n)$ of isomorphism classes of *framed sheaves* on $\mathbb{CP}^2$ of rank r with second Chern class n.

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A *framed sheaf* on $\mathbb{CP}^2$ of rank r with second Chern class n is a pair (E, α), in which

- E is a torsion free sheaf on $\mathbb{CP}^2$, a vector bundle in a neighborhood of a fixed line l_∞,
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Question

Is there a symplectic structure on $\mathcal{M}(r,n)$?

Answer

By using the ADHM data description, Nakajima realized the moduli space $\mathcal{M}(r,n)$ as a hyper-Kähler quotient.

By fixing a complex structure on $\mathcal{M}(r,n)$, one can define a holomorphic symplectic form on $\mathcal{M}(r,n)$.

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- Bottacin defined Poisson structures on moduli spaces of framed $\underline{vector}$ $\underline{bundles}$ on surfaces.

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He constructed a symplectic structure on $\mathcal{M}_{lf}(r,n)$ and on moduli spaces of framed $\underline{vector}$ $\underline{bundles}$ on other rational surfaces.

- By using a modified version of the Atiyah class for a family of framed sheaves, I defined closed two-forms on moduli spaces of framed $\underline{sheaves}$ on surfaces.

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I constructed a symplectic structure on moduli spaces of framed $\underline{sheaves}$ on some birationally ruled surfaces.

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Future directions of research

- Characterize the *Lagrangian subvarieties* of the moduli spaces of framed sheaves.

  In the case of $\mathcal{M}(r, n)$, the Lagrangian subvarieties parametrize solutions of *vortex equations*.
  (see Bonelli, Tanzini, Zhao. *Vertices, Vortices and Interacting Surface Operators* (arXiv:1102.0184)).

- Define an *algebraically integrable system* on the moduli spaces of framed sheaves , i.e., a proper flat morphism

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Warning!

In the following I will deal with schemes and coherent sheaves on them.

By using Serre's *GAGA principles*, one can think:

ALGEBRAIC GEOMETRY	$\Leftrightarrow$	COMPLEX DIFFERENTIAL GEOMETRY
Noetherian schemes of finite type over $\mathbb{C}$	$\Leftrightarrow$	Complex analytic spaces
Smooth varieties over $\mathbb{C}$	$\Leftrightarrow$	complex manifolds
Coherent sheaves	$\Leftrightarrow$	Coherent analytic sheaves

Let X be a smooth connected projective surface over $\mathbb{C}$.

Definition

Let D be an effective divisor of X and F_D a vector bundle on D.

We say that a coherent sheaf E on X is (D, F_D)-*framable* if

- E is torsion free,
- E is a vector bundle in a neighborhood of D,
- there is an isomorphism $E|_D \xrightarrow{\sim} F_D$.

An isomorphism $\alpha\colon E|_D \xrightarrow{\sim} F_D$ will be called a (D, F_D)-*framing* of E.

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A *(D, F_D)-framed sheaf* is a pair $\mathcal{E} := (E, \alpha)$ consisting of

- a (D, F_D)-framable sheaf E,
- a (D, F_D)-framing α.

Two (D, F_D)-framed sheaves (E, α) and (E', α') are isomorphic if there is an isomorphism $f\colon E \to E'$ such that $\alpha' \circ f|_D = \alpha$.

Framed sheaves

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Framed modules

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Theorem (Bruzzo, Markushevich)

There exists a moduli space $\mathcal{M}(X; F_D, P)$ for (D, F_D)-framed sheaves on X with Hilbert polynomial P, under the following assumptions:

- D is a big and nef divisor,
- F_D is a Gieseker semistable vector bundle on D.

$\mathcal{M}(X; F_D, P)$ is a quasi-projective scheme over $\mathbb{C}$.

If the surface X is rational and D is a smooth connected curve such that $D \cong \mathbb{CP}^1$ and $D^2 > 0$, $\mathcal{M}(X; \mathcal{O}_D^{\oplus r}, P)$ is a smooth quasi-projective variety.

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The Atiyah class

Let Y be a Noetherian scheme of finite type over $\mathbb{C}$.

Definition

Let E be a coherent sheaf on Y. We call *sheaf of first jets* $\mathrm{J}^1(E)$ of E the coherent sheaf on Y defined as follows:

- as a sheaf of $\mathbb{C}$-vector spaces, we set $\mathrm{J}^1(E) := (\Omega^1_Y \otimes E) \oplus E$,
- for any $y \in Y$, $a \in \mathcal{O}_{Y,y}$ and $(z \otimes e, f) \in \mathrm{J}^1(E)_y$, we define

$$a(z \otimes e, f) := (az \otimes e + d(a) \otimes f, af).$$

The sheaf $\mathrm{J}^1(E)$ fits into an exact sequence of coherent sheaves

$$0 \longrightarrow \Omega^1_Y \otimes E \longrightarrow \mathrm{J}^1(E) \longrightarrow E \longrightarrow 0. \tag{1}$$

Definition

Let E be a coherent sheaf on Y. We call *Atiyah class* of E the class $at(E)$ in $\mathrm{Ext}^1(E, \Omega^1_Y \otimes E)$ associated to the extension (1).

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Let E be a coherent sheaf on Y. An *algebraic connection* ∇ on E is a $\mathbb{C}$-linear morphism

$$\nabla\colon E \longrightarrow \Omega^1_Y \otimes E$$

such that (locally) $\nabla(f \cdot e) = f \cdot \nabla(e) + d(f) \otimes e$.

Proposition

The Atiyah class $at(E)$ is the obstruction to the existence of an algebraic connection on E,i.e., $at(E) = 0$ iff there exists an algebraic connection on E.

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Let S be a Noetherian scheme of finite type over $\mathbb{C}$.

Definition

A *flat family of (D, F_D)-framed sheaves parametrized by S* is a pair $\mathcal{E} = (E, \alpha)$ where

- E is a coherent sheaf on $S \times X$, flat over S,
- $\alpha\colon E \to p_X^*(F_D)$ is a morphism,

such that for any $s \in S$ the pair $(E|_{\{s\}\times X}, \alpha|_{\{s\}\times X})$ is a $(\{s\} \times D, p_X^*(F_D)|_{\{s\}\times D})$-framed sheaf on $\{s\} \times X$.

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To the coherent sheaf E, we associate its sheaf of first jets

$$\mathrm{J}^1(E) \cong (\Omega^1_{S\times X} \otimes E) \oplus E \text{ as sheaf of } \mathbb{C}\text{-vector spaces.}$$

Since $\Omega^1_{S\times X} \cong p_S^*(\Omega^1_S) \oplus p_X^*(\Omega^1_X)$, we have

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as sheaf of $\mathbb{C}$-vector spaces.

The *framed sheaf of first jets* $J^1_{fr}(\mathcal{E})$ is the subsheaf of the sheaf of first jets $\mathrm{J}^1(E)$ consisting of those sections whose $p_S^*(\Omega^1_S)$-part vanishes along $S \times D$.

To the coherent sheaf E, we associate its sheaf of first jets

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The framed sheaf of first jets $\mathrm{J}^1_{fr}(\mathcal{E})$ of $\mathcal{E}$ fits into an exact sequence:

$$0 \longrightarrow \big(p_S^*(\Omega_S^1)(-S\times D)\oplus p_X^*(\Omega_X^1)\big)\otimes E \longrightarrow \mathrm{J}^1_{fr}(\mathcal{E}) \longrightarrow E \longrightarrow 0, \tag{2}$$

where $p_S^*(\Omega_S^1)(-S\times D) := p_S^*(\Omega_S^1)\otimes \mathcal{O}_{S\times X}(-S\times D)$.

Definition

Let $\mathcal{E}=(E,\alpha)$ be a flat family of framed sheaves parametrized by a scheme S.
We call *framed Atiyah class* of the family $\mathcal{E}$ the class $at(\mathcal{E})$ in

$$\mathrm{Ext}^1\big(E, \big(p_S^*(\Omega_S^1)(-S\times D)\oplus p_X^*(\Omega_X^1)\big)\otimes E\big)$$

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The Kodaira-Spencer map for framed sheaves

The framed Atiyah class $at(\mathcal{E})$ of $\mathcal{E}$ induces a morphism $\mathcal{A}t_S(\mathcal{E})$

$$\mathcal{O}_S \to \mathcal{E}xt^1_{p_S}(E, p_S^*(\Omega^1_S) \otimes p_X^*(\mathcal{O}_X(-D)) \otimes E).$$

Definition

The *framed Kodaira-Spencer map* associated to the family $\mathcal{E}$ is the composition

$$\begin{aligned} KS_{fr} \colon (\Omega^1_S)^\vee &\xrightarrow{\mathrm{id}\otimes\mathcal{A}t_S(\mathcal{E})} \\ &\longrightarrow (\Omega^1_S)^\vee \otimes \mathcal{E}xt^1_{p_S}(E, p_S^*(\Omega^1_S) \otimes p_X^*(\mathcal{O}_X(-D)) \otimes E) \to \\ &\longrightarrow \mathcal{E}xt^1_{p_S}(E, p_S^*((\Omega^1_S)^\vee \otimes \Omega^1_S) \otimes p_X^*(\mathcal{O}_X(-D)) \otimes E) \to \\ &\longrightarrow \mathcal{E}xt^1_{p_S}(E, p_X^*(\mathcal{O}_X(-D)) \otimes E). \end{aligned}$$

The Kodaira-Spencer map for framed sheaves

The framed Atiyah class $at(\mathcal{E})$ of $\mathcal{E}$ induces a morphism $\mathcal{A}t_S(\mathcal{E})$

$$\mathcal{O}_S \to \mathcal{E}xt^1_{p_S}(E, p_S^*(\Omega^1_S) \otimes p_X^*(\mathcal{O}_X(-D)) \otimes E).$$

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Remark

Let S be a smooth projective variety over $\mathbb{C}$ and s a point on it. Then the framed Kodaira-Spencer map at the point s is

$$\begin{aligned} KS_{fr} \colon T_{S,s} &\to (\mathcal{E}xt^1_{p_S}(E, p_X^*(\mathcal{O}_X(-D)) \otimes E))_s \\ &\cong \operatorname{Ext}^1(E|_{\{s\}\times X}, E|_{\{s\}\times X}(-D)). \end{aligned}$$

Closed two-forms via the framed Atiyah class

Let $\mathcal{E} = (E, \alpha)$ be a flat family of framed sheaves parametrized by a smooth affine Noetherian scheme S of finite type over $\mathbb{C}$.

From the Atiyah class $at(\mathcal{E})$ of $\mathcal{E}$, we can define a class γ in

$$\mathrm{H}^0(S, \Omega_S^2) \otimes \mathrm{H}^2(X, \mathcal{O}_X(-2\,D)).$$

γ is the (0,2)-part of the *Newton polynomial* of $at(\mathcal{E})$.

Definition

Let τ_S be the homomorphism given by

$$\tau_S \colon \mathrm{H}^0(X, \omega_X(2\,D)) \cong \mathrm{H}^2(X, \mathcal{O}_X(-2\,D))^\vee \xrightarrow{\cdot\gamma} \mathrm{H}^0(S, \Omega_S^2),$$

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Remark

Fix $\omega \in \mathrm{H}^0(X, \omega_X(2\,D))$. The two-form $\tau_S(\omega)$ at a point $s_0 \in S$ coincides with the following composition of maps:

$$T_{s_0}S \times T_{s_0}S \stackrel{KS \times KS}{\longrightarrow}$$
$$\mathrm{Ext}^1(E|_{\{s_0\}\times X}, E|_{\{s_0\}\times X}(-D)) \times \mathrm{Ext}^1(E|_{\{s_0\}\times X}, E|_{\{s_0\}\times X}(-D))$$
$$\stackrel{\circ}{\longrightarrow} \mathrm{Ext}^2(E|_{\{s_0\}\times X}, E|_{\{s_0\}\times X}(-2\,D)) \stackrel{tr}{\longrightarrow} \mathrm{H}^2(X, \mathcal{O}_X(-2\,D))$$
$$\stackrel{\cdot\,\omega}{\longrightarrow} \mathrm{H}^2(X, \omega_X) \cong \mathbb{C}.$$

Proposition

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Let D be big and nef effective divisor, F_D a Gieseker semistable vector bundle on D and P a numerical polynomial of degree two.

$\mathcal{M}(X; F_D, P)$ = moduli space of (D, F_D)-framed sheaves on X with Hilbert polynomial P.

$\mathcal{M}(X; F_D, P)^{sm}$ = the smooth locus of $\mathcal{M}(X; F_D, P)$.

$\tilde{\mathcal{E}} = (\tilde{E}, \tilde{\alpha})$ = the universal family of $\mathcal{M}(X; F_D, P)^{sm}$.

Theorem

The framed Kodaira-Spencer map defined by $\tilde{\mathcal{E}}$ induces a canonical isomorphism

$$KS_{fr} \colon T\mathcal{M}(X; F_D, P)^{sm} \xrightarrow{\sim} \mathcal{E}xt^1_p(\tilde{E}, \tilde{E} \otimes p_X^*(\mathcal{O}_X(-D))).$$

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Tangent bundle of moduli spaces of framed sheaves

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Closed two-forms on moduli spaces of framed sheaves

Let $[(E,\alpha)] \in \mathcal{M}(X; F_D, P)^{sm}$. Then

$$T_{[(E,\alpha)]}\mathcal{M}(X; F_D, P) = \mathrm{Ext}^1(E, E(-D)).$$

For any $\omega \in \mathrm{H}^0(X, \omega_X(2\,D))$, we can define a skew-symmetric bilinear form

$$\mathrm{Ext}^1(E, E(-D)) \times \mathrm{Ext}^1(E, E(-D)) \xrightarrow{\circ} \mathrm{Ext}^2(E, E(-2\,D))$$
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By varying the point $[(E,\alpha)]$, these forms fit into an exterior two-form $\tau(\omega)$ on $\mathcal{M}(X; F_D, P)^{sm}$.

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Theorem

For any $\omega \in \mathrm{H}^0(X, \omega_X(2\,D))$, the two-form $\tau(\omega)$ on $\mathcal{M}(X; F_D, P)^{sm}$ is closed.

Proof. It suffices to prove that given a smooth affine scheme S, for any S-flat family $\mathcal{E} = (E, \alpha)$ of (D, F_D)-framed sheaves on X defining a classifying morphism

$$\begin{aligned} \psi\colon S &\longrightarrow \mathcal{M}(X; F_D, P)^{sm}, \\ s &\longmapsto [\mathcal{E}|_{\{s\}\times X}], \end{aligned}$$

the pullback $\psi^*(\tau(\omega)) \in \mathrm{H}^0(S, \Omega_S^2)$ is closed.

Since $\psi^*(\tau(\omega)) = \tau_S(\omega)$ by construction and $\tau_S(\omega)$ is closed, we get the assertion. □

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Non-degeneracy criterion for two-forms

Let $\omega \in \mathrm{H}^0(X, \omega_X(2\,D))$ and $[(E, \alpha)]$ a point in $\mathcal{M}(X; F_D, P)^{sm}$.

Proposition

The closed two-form $\tau(\omega)$ is non-degenerate at the point $[(E, \alpha)]$ if and only if the multiplication by ω induces an isomorphism

$$\omega_* \colon \mathrm{Ext}^1(E, E(-D)) \longrightarrow \mathrm{Ext}^1(E, E \otimes \omega_X(D)).$$

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If $\omega_X(2\,D)$ is trivial, then $1 \in \mathrm{H}^0(X, \omega_X(2\,D)) \cong \mathbb{C}$ defines a non-degenerate closed two-form.

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Definition

Let $p \in \mathbb{Z}, p \geq 0$. The p-th Hirzebruch surface is

$$\mathbb{F}_p := \mathbb{P}(\mathcal{O}_{\mathbb{CP}^1} \oplus \mathcal{O}_{\mathbb{CP}^1}(-p)).$$

Remark

- $\mathbb{F}_p$ is the projective closure of the total space of the line bundle $\mathcal{O}_{\mathbb{CP}^1}(-p)$ on $\mathbb{CP}^1$.
- $\mathbb{F}_p$ is the divisor in $\mathbb{CP}^2 \times \mathbb{CP}^1$

$$\mathbb{F}_p = \{([z_0 : z_1 : z_2], [z : w]) \in \mathbb{CP}^2 \times \mathbb{CP}^1 \mid z_1 w^p = z_2 z^p\}.$$

Examples on Hirzebruch surfaces

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Let $\pi\colon \mathbb{F}_p \to \mathbb{CP}^2$ be the projection onto $\mathbb{CP}^2$ and by l_∞ the inverse image of a generic line of $\mathbb{CP}^2$ through π.

Fact

l_∞ is a smooth connected big and nef curve of genus zero.

The Picard group of $\mathbb{F}_p$ is generated by l_∞ and the fibre F of the projection $\mathbb{F}_p \to \mathbb{CP}^1$. One has

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In particular, the canonical divisor K_p can be expressed as

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The Hirzebruch surface $\mathbb{F}_1$ is isomorphic to the blowup of $\mathbb{CP}^2$ at a point. Let D be a smooth connected curve of the complete linear system $|l_\infty + F|$.

Facts

- D is a smooth connected curve such that $D \cong \mathbb{CP}^1$ and $D^2 > 0$.
- $K_{\mathbb{F}_1} = -2\, l_\infty - F \;\Rightarrow \omega_{\mathbb{F}_1}(2D) \cong \mathcal{O}_{\mathbb{F}_1}(F)$.

Let $n \in \mathbb{Z}$ and F_D a Gieseker semistable vector bundle on D of rank r and degree $a+b$, for $a, b \in \mathbb{Z}$.

$\mathcal{M}(\mathbb{F}_1; F_D, r, a\, l_\infty + b\, F, n)$ = the moduli space of (D, F_D)-framed sheaves on $\mathbb{F}_1$ of rank r, first Chern class $a\, l_\infty + b\, F$ and second Chern class n.

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Let l be a smooth connected curve of genus zero in $\mathbb{F}_1$ defined by a nonzero section $\omega_l \in \mathrm{H}^0(\mathbb{F}_1, \omega_{\mathbb{F}_1}(2\,D))$.

$\mathcal{M}_{lf}(\mathbb{F}_1; F_D, r, a\,l_\infty + b\,F, n)$ = open subscheme of $\mathcal{M}(\mathbb{F}_1; F_D, r, a\,l_\infty + b\,F, n)$ consisting of points $[(E, \alpha)]$ with E trivial along l.

Theorem

The two-form $\tau(\omega_l)$ defines a holomorphic symplectic structure on the smooth locus of $\mathcal{M}_{lf}(\mathbb{F}_1; F_D, r, a\,l_\infty + b\,F, n)$.

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Let $\mathbb{F}_2$ be the second Hirzebruch surface. It is the projective closure of the cotangent bundle $T^*\mathbb{CP}^1$ of the complex projective line $\mathbb{CP}^1$.

Fact:

- The canonical divisor $K_{\mathbb{F}_2} = -2\, l_\infty \Rightarrow \omega_{\mathbb{F}_2}(2\, l_\infty)$ is trivial.

Let $D = l_\infty$, $n \in \mathbb{Z}$ and F_D a Gieseker semistable vector bundle on D of rank r and degree $2a + b$, with $a, b \in \mathbb{Z}$.

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The two-form $\tau(1)$ *defines a symplectic structure on* $\mathcal{M}(\mathbb{F}_2; F_D, r, a\, l_\infty + b\, F, n)^{sm}$.

If $F_D \cong \mathcal{O}_D^{\oplus r}$, we have $b = -2a$.

Let us define $C = l_\infty - 2F$. This is the only irreducible curve in $\mathbb{F}_2$ with negative self intersection. We can normalize the value a in the range $0 \leq a \leq r - 1$ upon twisting by $\mathcal{O}_{\mathbb{F}_2}(C)$.

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