Flux Compactifications and Matrix Models for Superstrings

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Based on:

A.C., 1108.1107 [hep-th] (PRD 84 (2011))
A.C. and Larisa Jonke, 1202.4310 [hep-th] (PRD 85 (2012))
A.C. and Larisa Jonke, 1207.6412 [hep-th]

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14.11.12

Introduction and Motivation

Main Objective

Study properties of string compactifications beyond low-energy sugra.

Mainly, unconventional compactifications
$\rightsquigarrow$ related to string length, not captured by vanilla sugra
(winding modes, dualities, non-geometric fluxes, non-commutative manifolds etc.).

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Frameworks:

- Doubled formalism - Twisted Doubled Tori
- Generalized Complex Geometry
- Double Field Theory
- CFT - Sigma models
- ✔ Matrix Models

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Frameworks:

- Doubled formalism - Twisted Doubled Tori Hull; Hull, Reid-Edwards; Dall'Agata et.al.
- Generalized Complex Geometry Andriot et.al.; Berman et.al.
- Double Field Theory
  Hohm, Hull, Zwiebach; Aldazabal et.al.; Geissbuhler; Grana, Marques; Dibitetto et.al.
- CFT - Sigma models Lüst; Blumenhagen, Plauschinn; Mylonas, Schupp, Szabo
- ✔ Matrix Models Lowe, Nastase, Ramgoolam; A.C., Jonke

Why Matrix Models?

Advantages:

- ✔ Non-perturbative framework.
- ✔ Non-commutative structures.
- ✔ Quantization.
- ✔ Possible phenomenological applications
  - Particle physics, "matrix model building". Aoki '10-'12, A.C., Steinacker, Zoupanos '11
  - Early and late time cosmology. Kim, Nishimura, Tsuchiya '11-'12

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Disadvantages:

- × Sugra limit is not clear.
- × Less calculability.

Matrix Models as non-perturbative definitions of string/M theory.

Banks, Fischler, Shenker, Susskind '96, Ishibashi, Kawai, Kitazawa, Tsuchiya '96, ...

Matrix Model Compactifications (MMC) on non-commutative tori.

Connes, Douglas, A. Schwarz '97

Constant background B-field $\longleftrightarrow$ Non-commutative deformation

$$B_{ij} \quad \overset{CDS}{\longleftrightarrow} \quad \theta^{ij}$$

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What about fluxes?

- Geometric (related e.g. to nilmanifolds/twisted tori): f
- NSNS (e.g. non-constant B-fields): H
- "Non-geometric" (T-duality): Q, R

Q: How can they be traced in Matrix Compactifications?

Overview

Matrix Models

IKKT: non-perturbative IIB superstring, Ishibashi, Kawai, Kitazawa, Tsuchiya '96

$$Z = \int d\mathcal{X} d\Psi e^{-S},$$

with action

$$\mathcal{S}_{IKKT} = \frac{1}{2g}\mathrm{Tr}\Big(-\frac{1}{2}[\mathcal{X}_a, \mathcal{X}_b]^2 - \bar{\Psi}\Gamma^a[\mathcal{X}_a, \Psi]\Big).$$

$\mathcal{X}_a$: 10 $N \times N$ Hermitian matrices (large N); Ψ: fermionic superpartners.

BFSS: non-perturbative M-theory, Banks, Fischler, Shenker, Susskind '96

$$\mathcal{S}_{BFSS} = \frac{1}{2g}\int dt\Big[Tr\big(\dot{\mathcal{X}}_a\dot{\mathcal{X}}_a - \frac{1}{2}[\mathcal{X}_a, \mathcal{X}_b]^2\big) + \text{fermions}\Big],$$

$\mathcal{X}_a(t)$: 9 and time-dependent...

Classical solutions

EOM (IKKT; setting $\Psi = 0$):

$$\sum_b [\mathcal{X}_b, [\mathcal{X}_b, \mathcal{X}_a]] = 0.$$

- Basic solutions:

$$[\mathcal{X}_a, \mathcal{X}_b] = i\theta_{ab}$$

  $\text{Rank}(\theta) = p + 1 \Rightarrow$ Dp brane.

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- Lie algebra type?

  $$[\mathcal{X}_a, \mathcal{X}_b] = if_{ab}^{\ \ c}\mathcal{X}_c$$

  If no deformation $\rightsquigarrow$ no semisimple. Nilpotent and solvable?
  Fully classified up to 7D (6D: finite) Morozov '58, Mubarakzyanov '63, Patera et.al. '75
  Resulting solutions: 7 nilpotent (3D, 5D(2), 6D(4)) + 2 solvable (4D, 5D).
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Why is this interesting?

✔ Play role in cosmological studies based on IKKT. Kim, Nishimura, Tsuchiya '11-'12

✔ Starting point for a class of compact manifolds (nil- and solvmanifolds).

Nilmanifolds Mal'cev '51

Smooth manifolds $\mathcal{M} = G/\Gamma$
G: Nilpotent Lie group; Γ: Discrete co-compact subgroup of G.

Nilpotency $\rightsquigarrow$ upper triangular matrices...

Construction algorithm:

α. Find a basis T_a of Lie(G) in terms of upper triangular matrices.

β. Choose a representative group element $g \in G$.

γ. Define the restriction of g for integer matrix entries ($\gamma \in \Gamma$).

δ. Γ acts on G by matrix multiplication. Quotient out this action and construct G/Γ.

Some geometry

Lie algebra 1-form $e = g^{-1}dg = e^a T_a$.

e^a correspond to the vielbein basis and there is a twist matrix such that:

$$e^a = U(x)^a_b dx^b$$

They satisfy the Maurer-Cartan equations

$$de^a = -\frac{1}{2} f^a{}_{bc} e^b \wedge e^c,$$

$f^a{}_{bc}$ being the structure constants of Lie(G) $\sim$ geometric fluxes.

Certain periodicity conditions render e^a globally well-defined. Thus nilmanifolds are (iterated) twisted fibrations of toroidal fibers over toroidal bases.

$$
\begin{array}{ccc}
T^{D_n} & \hookrightarrow & \mathcal{M}_{\sum_i D_i} \\
 & & \downarrow \\
 & & \vdots \\
 & & \downarrow \\
T^{D_3} & \hookrightarrow & \mathcal{M}_{D_1+D_2+D_3} \\
 & & \downarrow \\
T^{D_2} & \hookrightarrow & \mathcal{M}_{D_1+D_2} \\
 & & \downarrow \\
 & & T^{D_1}
\end{array}
$$

$\rightarrow$ The number of such iterations is set by the nilpotency class.

Prototype example: 3D

3D nilpotent Lie algebra: $[T_1, T_2] = T_3$.

Upper triangular basis:

$$T_1 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad T_2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}, \quad T_3 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Group element: $g = \begin{pmatrix} 1 & x^1 & x^3 \\ 0 & 1 & x^2 \\ 0 & 0 & 1 \end{pmatrix}, x^i \in \mathbb{R}$.

Restriction to Γ: $g|_\Gamma = \begin{pmatrix} 1 & \gamma^1 & \gamma^3 \\ 0 & 1 & \gamma^2 \\ 0 & 0 & 1 \end{pmatrix}, \gamma^i \in \mathbb{Z}$.

Invariant 1-form: $e = \begin{pmatrix} 0 & dx^1 & dx^3 - x^1 dx^2 \\ 0 & 0 & dx^2 \\ 0 & 0 & 0 \end{pmatrix}$.

Its components are: $e^1 = dx^1, \quad e^2 = dx^2, \quad e^3 = dx^3 - x^1 dx^2.$

Twist matrix: $U = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -x^1 & 1 \end{pmatrix}$.

Reading off the required identifications:

$$(x^1, x^2, x^3) \sim (x^1, x^2+2\pi R_2, x^3) \sim (x^1, x^2, x^3+2\pi R_3) \sim (x^1+2\pi R_1, x^2, x^3+2\pi R_1 x^2)$$

$$\begin{array}{ccc} T^2_{(2,3)} & \hookrightarrow & \mathcal{M} = \tilde{T}^3 \\ & & \downarrow \\ & & S^1_{(1)} \end{array}$$

T-duality approach

Alternatively, consider a square torus with N units of NSNS flux $H = dB$, proportional to its volume form:

- ✔ Metric: $ds^2 = \delta_{ab} dx^a dx^b$.
- ✔ B-field: $B_{23} = Nx^1$.

Perform a T-duality along x^3 using the Buscher rules:

$$G_{ii} \xrightarrow{T_i} \frac{1}{G_{ii}}, \quad G_{ai} \xrightarrow{T_i} \frac{B_{ai}}{G_{ii}}, \quad G_{ab} \xrightarrow{T_i} G_{ab} - \frac{G_{ai}G_{bi} - B_{ai}B_{bi}}{G_{ii}},$$
$$B_{ai} \xrightarrow{T_i} \frac{G_{ai}}{G_{ii}}, \quad B_{ab} \xrightarrow{T_i} B_{ab} - \frac{B_{ai}G_{bi} - G_{ai}B_{bi}}{G_{ii}}$$

In the T-dual frame:

- ✔ Metric: $ds^2 = \delta_{ab} e^a e^b \rightsquigarrow e^a$ of $\tilde{T}^3$.
- ✔ B-field: $B = 0$.

Depicted as:

$$H_{abc} \xleftrightarrow{T_c} f_{ab}^{\ c}$$

Matrix Model Compactification-Tori

Connes, Douglas, Schwarz '97

Restriction of the action functional under periodicity conditions.

Toroidal T^d:

$$\begin{aligned} U^i \mathcal{X}_i (U^i)^{-1} &= \mathcal{X}_i + 1, \quad i = 1, ..., d, \\ U^i \mathcal{X}_a (U^i)^{-1} &= \mathcal{X}_a, \quad a \neq i, \quad a = 1, \ldots, 9, \end{aligned}$$

with U^i unitary and invertible (gauge transformations of the model).

Solutions Connes, Douglas, Schwarz '97

$$\mathcal{X}_i = iR_i\hat{\mathcal{D}}_i, \quad \mathcal{X}_m = \mathcal{A}_m(\hat{U}), (m = d+1, \ldots, 9), \quad U^i = e^{i\hat{x}^i},$$

with covariant derivatives $\hat{\mathcal{D}}_i = \hat{\partial}_i - i\mathcal{A}_i(\hat{U})$.

The U-algebra is in general: $U^i U^j = \lambda^{ij} U^j U^i$ with complex constants $\lambda^{ij} = e^{-i\theta^{ij}}$.
$\rightsquigarrow$ non-commutative torus. Connes, Rieffel

$\mathcal{A}$'s depend on a set of operators $\hat{U}$, commuting with U: Brace, Morariu, Zumino '98

$$\hat{U}_i = e^{i\hat{x}^i - \theta^{ij}\hat{\partial}_j},$$

satisfying dual relations $\hat{U}_i\hat{U}_j = e^{i\hat{\theta}^{ij}}\hat{U}_j\hat{U}_i, \quad \hat{\theta}^{ij} = -\theta^{ij}$.

Substitution back into the action $\rightsquigarrow$ NCSYM theory on the dual NC torus.

Note: the solution involves a quantized phase space of $\hat{x}$ and $\hat{p}$ with algebra:

$$\begin{aligned} [\hat{x}^i, \hat{x}^j] &= i\theta^{ij}, \\ [\hat{x}^i, \hat{p}_j] &= i\delta^i_j, \\ [\hat{p}_i, \hat{p}_j] &= 0. \end{aligned}$$

Interpretation: Deformation parameters θ correspond to moduli of a sugra compactification, i.e. they are reciprocal to a background B field,

$$(\theta^{-1})_{ij} \propto \int dx^i dx^j B_{ij}$$

Matrix Model Compactification-Nilmanifolds

Restrict the action by imposing conditions corresponding to nilmanifolds.
Lowe, Nastase, Ramgoolam '03; A.C., Jonke '11-'12

3D nilmanifold $\tilde{T}^3$ (in a more "democratic gauge"):

$$\begin{aligned} U^i \mathcal{X}_i (U^i)^{-1} &= \mathcal{X}_i + 1, \quad i = 1,2,3, \\ U^1 \mathcal{X}_3 (U^1)^{-1} &= \mathcal{X}_3 - \mathcal{X}_2, \quad U^2 \mathcal{X}_3 (U^2)^{-1} = \mathcal{X}_3 + \mathcal{X}_1, \\ U^i \mathcal{X}_a (U^i)^{-1} &= \mathcal{X}_a, \quad a \neq i, \quad a = 1,\dots,9, \quad (a,i) \neq \{(3,1),(3,2)\}. \end{aligned}$$

Solutions:

$$\mathcal{X}_i = iR_i \hat{\mathcal{D}}_i, \quad \mathcal{X}_m = \mathcal{A}_m(\hat{U}), (m = 4,\dots,9), \quad U^i = e^{i\hat{x}^i},$$

with covariant derivatives $\hat{\mathcal{D}}_i = \hat{\partial}_i - i\mathcal{A}_i(\hat{U}) + f_i{}^{jk}\mathcal{A}_j(\hat{U})\hat{\partial}_k, \quad f_3{}^{12} \neq 0.$

The U-algebra is now given by: $U^i U^j = e^{-i\theta^{ij} - if^{ij}{}_k \hat{x}^k} U^j U^i$.

$\rightsquigarrow$ non-commutative twisted torus Lowe, Nastase, Ramgoolam '03; A.C., Jonke '12; c.f. Rieffel '89

The dual operators are now $\hat{U} = e^{i\hat{y}^i}$ with: $\hat{y}^i = \hat{x}^i - i\theta^{ij}\hat{\partial}_j - if^{ij}{}_k \hat{x}^k \hat{\partial}_j$.
Algebra of phase space:

$$
\begin{aligned}
[\hat{x}^i, \hat{x}^j] &= i\theta^{ij} + if^{ij}{}_k \hat{x}^k \equiv i\theta^{ij}(\hat{x}), \\
[\hat{p}_i, \hat{x}^j] &= -i\delta_i^j - if_i{}^{jk}\hat{p}_k, \\
[\hat{p}_i, \hat{p}_j] &= 0.
\end{aligned}
$$

The effective action is a NC gauge theory on a dual NC twisted torus.

Interpretation: The non-constant deformation is the analog of a geometric flux.

Direct generalization for all higher-D nilmanifolds, richer in geometric fluxes.

More fluxes?

At hand: geometric flux ${f_{ij}}^k$ (nilmanifold).

T-dual to NSNS flux H_{ijk}: $\quad H_{ijk} \stackrel{T_k}{\longleftrightarrow} {f_{ij}}^k$.

Enlarged chain with unconventional fluxes:

$$H_{ijk} \stackrel{T_k}{\longleftrightarrow} f_{ij}{}^{k} \stackrel{T_j}{\longleftrightarrow} Q_i{}^{jk} \stackrel{T_i}{\longleftrightarrow} R^{ijk}.$$

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or

Q: Which compactifications correspond to more general phase space algebras?

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Q: Matrix Model description?

or

Q: Which compactifications correspond to more general phase space algebras?

or

Q: What is the role of, previously ignored, $\tilde{U}_i = e^{i\hat{p}_i}$ (esp. when $[\hat{p}, \hat{p}] \neq 0$)?

Building Blocks

H-block: Consider the phase space algebra c.f. Lüst '10:

$$
\begin{aligned}
[\hat{x}^i, \hat{x}^j] &= iF^{ijk}\hat{p}_k, \\
[\hat{x}^j, \hat{p}_i] &= i\delta_i^j, \\
[\hat{p}_i, \hat{p}_j] &= 0.
\end{aligned}
$$

If $U^i = e^{i\hat{x}^i}$ and $\tilde{U}_i = e^{i\hat{p}_i}$, and we make the Ansatz $\mathcal{X}_i = i\hat{D}_i$,

$$
\begin{aligned}
U^i \mathcal{X}_i (U^i)^{-1} &= \mathcal{X}_i + 1, \\
\tilde{U}_i \mathcal{X}_i (\tilde{U}_i)^{-1} &= \mathcal{X}_i,
\end{aligned}
$$

$\rightsquigarrow$ looks like familiar compactification on torus.

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$$\begin{aligned} U^i \mathcal{X}_i (U^i)^{-1} &= \mathcal{X}_i + 1, \\ \tilde{U}_i \mathcal{X}_i (\tilde{U}_i)^{-1} &= \mathcal{X}_i, \end{aligned}$$

$\rightsquigarrow$ looks like familiar compactification on torus.

BUT, the U-algebra is: $U^i U^j = e^{i\theta^{ij}(\hat{p})} U^j U^i$, with $\theta^{ij} = F^{ijk}\hat{p}_k$.

The Connes-Douglas-Schwarz correspondence suggests a sugra B-field

$$B = x^1 dx^2 \wedge dx^3 + x^2 dx^3 \wedge dx^1 + x^3 dx^1 \wedge dx^2 \quad \rightsquigarrow H_{ijk}$$

where x^i are standard toroidal coordinates.

The present algebra is related to the f-block one by a "canonical transformation":

$$
\begin{aligned}
\hat{x}^3 &\rightarrow -\hat{p}_3, \\
\hat{p}_3 &\rightarrow \hat{x}^3.
\end{aligned}
$$

Represent this as a matrix $M_{H\rightarrow f}$ acting on $\begin{pmatrix} \hat{x}^i \\ \hat{p}_i \end{pmatrix}$. The f-solution is mapped to the H-solution under the combined action of $M_{H\rightarrow f}$ and a grading correction

$$(-1)_f^{\hat{c}_i} = \mathrm{diag}(1,1,1,1,1,-1).$$

For the 3D case, this is depicted as:

$$
\begin{array}{ccc}
H & \overset{T_3}{\longleftrightarrow} & f \\
\updownarrow & & \updownarrow \\
\theta(\hat{p}) & \overset{M_{H\rightarrow f}\cdot(-1)_f^{\hat{c}_i}}{\longleftrightarrow} & \theta(\hat{x})
\end{array}
$$

Q-block: Consider a different phase space algebra:

$$
\begin{aligned}
[\hat{x}^i, \hat{x}^j] &= 0, \\
[\hat{p}_i, \hat{x}^j] &= -i\delta_i^j + iF_{ik}{}^j \hat{x}^k, \\
[\hat{p}_i, \hat{p}_j] &= -iF_{ij}{}^k \hat{p}_k.
\end{aligned}
$$

This is motivated by a transformation $M_{f \to Q}$ on $\hat{x}^2, \hat{p}_2$.

If $U^i = e^{i\hat{x}^i}$ and $\tilde{U}_i = e^{(-1)^{c_i} i\hat{p}_i}$ (with $c_1 = 1, c_{2,3} = 0$), the Ansatz $\mathcal{X}_i = i\hat{D}_i$ gives e.g.

$$
U^1 \mathcal{X}_2 (U^1)^{-1} = \mathcal{X}_2 - \hat{x}^3,
$$

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Two ways out:

- Introduce dual elements $\tilde{\mathcal{X}}^i \propto \hat{x}^i$, a kind of doubled formalism.
  This fits well with Twisted Doubled Tori approach to non-geometry
  Hull, Reid-Edwards '07,'09; Dall'Agata, Prezas, Samtleben, Trigiante '07
- Change the Ansatz.

Different Ansatz:

$$\mathcal{X}_i = i\delta_{ij}\hat{\tilde{D}}^j,$$

with $\hat{\tilde{D}}^i|_{\mathcal{A}=0} = (-1)^{c_i}\hat{\tilde{\partial}}^i$, where $i\hat{\tilde{\partial}}^i = i\frac{\partial}{\partial p_i}$ is the position in the momentum rep.

Then:

$$\begin{aligned} U^i\mathcal{X}_i(U^i)^{-1} &= \mathcal{X}_i, \\ \tilde{U}_i\mathcal{X}_i(\tilde{U}_i)^{-1} &= \mathcal{X}_i + 1, \\ \tilde{U}_2\mathcal{X}_1(\tilde{U}_2)^{-1} &= \mathcal{X}_1 - \mathcal{X}_3, \\ \tilde{U}_3\mathcal{X}_1(\tilde{U}_3)^{-1} &= \mathcal{X}_1 + \mathcal{X}_2, \end{aligned}$$

The U-algebra is commutative.
But the $\tilde{U}$ one is not: $\tilde{U}_i\tilde{U}_j = e^{i\tilde{\theta}_{ij}(\hat{p})}\tilde{U}_j\tilde{U}_i$, with $\tilde{\theta}_{ij} = -F_{ij}^k\hat{p}_k$.

What does this compactification correspond to?

Comparison to TDT approach; matches with a polarization of a T-fold with a Q-flux.

Alternatively: the algebra is obtained by an M-transformation on $\hat{x}^2, \hat{p}_2$.
This can be understood as a generalized T-duality Hull; Hull, Reid-Edwards

$$
\begin{array}{ccccc}
H & \xleftrightarrow{T_3} & f & \xleftrightarrow{T_2} & Q \\
\updownarrow & & \updownarrow & & \updownarrow \\
\theta(\hat{p}) & \xleftrightarrow{M_{H\to f}\cdot(-1)_f^{\hat{c}_i}} & \theta(\hat{x}) & \xleftrightarrow{M_{f\to Q}\cdot(-1)_Q^{\hat{c}_i}} & \tilde{\theta}(\hat{p})
\end{array}
$$

with $\mathrm{i}\theta^{ij} = [\hat{x}^i, \hat{x}^j]$ and $\mathrm{i}\tilde{\theta}_{ij} = [\hat{p}_i, \hat{p}_j]$.

R-block: In a similar spirit:

$$
\begin{aligned}
[\hat{x}^i, \hat{x}^j] &= 0, \\
[\hat{p}_i, \hat{x}^j] &= -i\delta_i^j, \\
[\hat{p}_i, \hat{p}_j] &= iF_{ijk}\hat{x}^k.
\end{aligned}
$$

Obtained from the previous via a $M_{Q\to R}$ on $\hat{x}^3, \hat{p}_3$.

Following the Ansatz of the previous case:

$$
\begin{aligned}
U^i \mathcal{X}_i (U^i)^{-1} &= \mathcal{X}_i, \\
\tilde{U}_i \mathcal{X}_i (\tilde{U}_i)^{-1} &= \mathcal{X}_i + 1,
\end{aligned}
$$

The Us commute again, unlike the $\tilde{U}$s: $\tilde{U}_i \tilde{U}_j = e^{-i\tilde{\theta}_{ij}(\hat{x})} \tilde{U}_j \tilde{U}_i$ with $\tilde{\theta}_{ij} = -F_{ijk}\hat{x}^k$.

Comparison with TDT approach, and within the generalized T-duality interpretation of $M \rightsquigarrow$ matches with a compactification with R flux.

Full Picture:

$$
\begin{array}{ccccccc}
H & \overset{T_3}{\longleftrightarrow} & f & \overset{T_2}{\longleftrightarrow} & Q & \overset{T_1}{\longleftrightarrow} & R \\
\updownarrow & & \updownarrow & & \updownarrow & & \updownarrow \\
\theta(\hat{p}) & \overset{M_{H\to f}\cdot(-1)_f^{\hat{c}_i}}{\longleftrightarrow} & \theta(\hat{x}) & \overset{M_{f\to Q}\cdot(-1)_Q^{\hat{c}_i}}{\longleftrightarrow} & \tilde{\theta}(\hat{p}) & \overset{M_{Q\to R}\cdot(-1)_R^{\hat{c}_i}}{\longleftrightarrow} & \tilde{\theta}(\hat{x})
\end{array}
$$

with $\quad \mathrm{i}\theta^{ij} = [\hat{x}^i, \hat{x}^j] \quad$ and $\quad \mathrm{i}\tilde{\theta}_{ij} = [\hat{p}_i, \hat{p}_j]$.

$\rightsquigarrow$ There is a correspondence:

$$\theta^{ij}|_f \quad \text{or} \quad \theta^{ij}|_H \quad \text{in} \quad \hat{x}\text{-space} \quad \longleftrightarrow \quad \tilde{\theta}_{ij}|_Q \quad \text{or} \quad \tilde{\theta}_{ij}|_R \quad \text{in} \quad \hat{p}\text{-space} \; .$$

- In position space: MMC with non-constant $\theta \sim$ geometric fluxes.
- In momentum space: MMC with non-constant $\tilde{\theta} \sim$ non-geometric fluxes.

Similar result in Generalized Complex Geometry approach...
Andriot, Larfors, Lüst, Patalong '11

Indication: Just as $\theta^{ij} \sim (B_{ij})^{-1}$, also $\tilde{\theta}_{ij} \sim (\beta^{ij})^{-1}$, β: the bivector of GCG.

It would be interesting to explore further such relations.

Non-Associativity and Flux Quantization

All encountered phase space algebras exhibit some non-associativity.

E.g. $[\hat{p}_i, \hat{x}^j, \hat{x}^k] \propto f_i{}^{jk}$ for the f-block, $[\hat{x}^i, \hat{x}^j, \hat{x}^k] \propto F^{ijk}$ for the H-block, etc.

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$\mathcal{X}_i$: They associate in all cases (in the regime where the compactification is well-defined).

U^i: e.g. in H-case: $U^i(U^j U^k) = e^{\frac{i}{2}H^{ijk}}(U^i U^j)U^k$.

⇝ 3-cocycle; typical in QM systems with fluxes. Jackiw '85

Resolution: The flux **has to** be quantized,

$$H = 4\pi n, \quad n \in \mathbb{Z}.$$

⇝ Flux Quantization is already built-in.

Flux Coexistence work in progress

In sugra, metric and NSNS fluxes can coexist.
Straightforward implementation in the MMC.

Q: Coexistence of all flux types, including non-geometric?

In our approach, two ways:

- ✔ Start with an appropriately rich pure geometry; find frame with all fluxes.
- ✔ Combine MM solutions block-diagonally.

First approach work in progress

Richer chain of duality frames:

$$\left\{\begin{array}{c} f_{i_1j_1}{}^{k_1} \\ f_{i_2j_2}{}^{k_2} \\ f_{i_3j_3}{}^{k_3} \\ f_{i_4j_4}{}^{k_4} \end{array}\right\} \overset{T_i}{\longleftrightarrow} \quad \dots \quad \overset{T_j}{\longleftrightarrow} \left\{\begin{array}{c} H_{i'_1j'_1k'_1} \\ f_{i'_2j'_2}{}^{k'_2} \\ Q_{i'_3}{}^{j'_3k'_3} \\ R^{i'_4j'_4k'_4} \end{array}\right\}.$$

If the simple chain is understood $\Rightarrow$ this is equally well understood.

In fact, up to mild requirements, there is a unique nilmanifold able to realize this,

$$\begin{array}{ccc} S^1_{(6)} & \hookrightarrow & \mathcal{M}_6 \\ & & \downarrow \\ T^2_{(4,5)} & \hookrightarrow & \mathcal{M}_5 \\ & & \downarrow \\ & & T^3_{(1,2,3)} \end{array}$$

Second approach work in progress

Solutions of MM can be combined block-diagonally.

Is it possible to use this property to define a MMC with solution e.g.

$$\mathcal{X}_i = \begin{pmatrix} \mathcal{X}_i^{(H)} & 0 \\ 0 & \mathcal{X}_i^{(R)} \end{pmatrix} \ ?$$

Which are the properties of such a MMC?
Are there some associated bound states?

Main messages

- ✔ Matrix Models: useful framework for unconventional string compactifications.
- ✔ Fluxes, dualities, non-geometry, non-commutativity.
- ✔ Relations to other frameworks (double field theory, generalized geometry, etc.)

Some prospects

- Analysis of the effective theories with fluxes. in progress, with L. Jonke
- Full study of possible vacua. Coexistence of all types of fluxes.
  in progress, with L. Jonke and M. Schmitz
- Phenomenology of unconventional compactifications?
- Non-perturbative dualities?