Quantization of non-geometric flux backgrounds

Dionysios Mylonas

School of Mathematical and Computer Science
Heriot-Watt university, Edinburgh

EMPG
November 14, 2012

Based on:

D.M, P. Schupp and R. Szabo, *JHEP* **1209** (2012) 012, [arXiv:1207.0926]

Outline

- The past: Background and motivation.
- From a membrane σ-model to string theory on the boundary.
- Deformation quantization a lá Kontsevich.
- Convolution product quantization.
- Recap.
- The future: ?

This is the end...

You have...
36d 9hr and 25min
left to the **END OF THE WORLD!**

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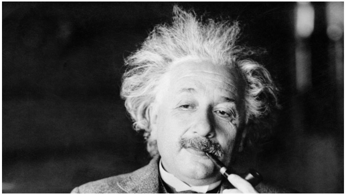

"Two things are infinite: the universe and human stupidity; and I'm not sure about the the universe."

Open and closed strings

Open string with constant B-field

$$\big[X^i(\tau,\sigma), X^j(\tau,\sigma')\big]_{\sigma=\sigma'=0,2\pi} = \mathrm{i}\,\theta^{ij}$$

where $\theta = -2\pi\,\alpha'\,(1+\mathcal{F}^2)^{-1}\,\mathcal{F}$ and $\mathcal{F} = B - F$.

(Seiberg & Witten, Chu & Ho)

- Commutator not well defined for closed strings. Jacobiator 3-bracket:

$$[X^i, X^j, X^k] := \lim_{\sigma_i\to\sigma}\, \Big[\big[X^i(\tau,\sigma_1), X^j(\tau,\sigma_2)\big], X^k(\tau,\sigma_3)\Big] + \text{cyclic}$$

- For the linearized $SU(2)$ WZW model with $H = dB \neq 0$ $[X^i, X^j, X^k] \sim H^{ijk}$, i.e. the target space is **non-associative**.

(Blumenhagen & Plauschinn, 1010.1263)

T-duality frames

Consider $\mathbb{T}^3$ with non-vanishing H-flux.

$$H\text{-flux} \xrightarrow{T_{x^3}} f\text{-flux} \xrightarrow{T_{x^2}} Q\text{-flux} \xrightarrow{T_{x^1}} R\text{-flux}$$

(nilmanifold) (T-fold) (non-geometric)

$$\begin{aligned} \left[X^i(\tau,\sigma), X^j(\tau,\sigma')\right] &= 0 \\ \left[X^i(\tau,\sigma), X^*_j(\tau,\sigma')\right] &\neq 0 \end{aligned} \quad \xrightarrow{T_{x^2}} \quad \begin{aligned} \left[X^i(\tau,\sigma), X_j(\tau,\sigma')\right] &\neq 0 \\ \left[X^i(\tau,\sigma), X^*_j(\tau,\sigma')\right] &= 0 \end{aligned}$$

where $X^*_i = X_{i,L} - X_{i,R} \in M^*$. *Should use doubled geometry.*

- In this context the same type of nonassociativity was found and a **NA target space algebra** on R-space was proposed.

(Lüst, 1010.1361)

Boundary conditions

- Closed str on $M \times M^*$

$$\begin{aligned} X(\tau, \sigma + 2\pi) &= \mathrm{e}^{2\pi\,\mathrm{i}\,\theta}\, X(\tau, \sigma) \\ X^*(\tau + 2\pi, \sigma) &= \mathrm{e}^{2\pi\,\mathrm{i}\,\theta}\, X^*(\tau, \sigma) \end{aligned}$$

  where $\theta = -n\,H$ and $n \in \mathbb{Z}$ the dual momentum along the T-dualised direction.

- The situation resembles the open string case:
  NCFT on D-branes $\xrightarrow{T}$ FT on intersecting D-branes
- Are there some kind of closed string "D-branes"? Are they dynamical solutions in doubled gravity? NA SW map?
  (Lüst, 1010.1361)

Linearized CFT

Flat space with constantl H-flux

$$[X^i, X^j, X^k] = \mathrm{i}\,\alpha\,\theta^{ijk}$$

where $\theta^{ijk} \sim H$.
$\alpha = 0$ for the H-flux background
$\alpha = 1$ after an odd number of T-duality transformations.

3-product conjecure for constant R-flux

$$(f_1 \diamond f_2 \diamond f_3)(x) := \exp\left(\frac{\pi^2}{2}\theta^{ijk}\partial_i^{x_1}\partial_j^{x_2}\partial_k^{x_3}\right) f_1(x_1) f_2(x_2) f_3(x_3)\Big|_{x_i = x}$$

(Blumenhagen et al, 1106.0316)

Definitions

Poisson manifold

Given a bivector field $\Theta = \frac{1}{2}\,\Theta^{ij}(x)\,\partial_i \wedge \partial_j$ on a smooth manifold $\mathcal{M}$ a skew-symmetric bracket $\{-,-\}_\Theta$ can be defined. This is a Poisson structure if the Schouten-Nijehuis bracket $[\Theta,\Theta]_S$ is zero.

A *quasi-Poisson structure* doesn't satisfy the Jacobi identity.

Lie algebroid

A *Lie algebroid* is a vector bundle $E \to \mathcal{M}$ endowed with a Lie bracket $[-,-]_E$ on smooth sections of E and an *anchor map* $\rho : E \to T\mathcal{M}$. The tangent map to ρ is a Lie algebra homomorphism.

Courant algebroid: E is further equipped with a metric $\langle -,-\rangle$ and a Jacobiator.

Poisson σ-model

- A Poisson manifold M ($=$ symplectic Lie 1-algebroid with the canonical symplectic structure on T^*M).
- A 2d string worldsheet Σ_2.
- A differential form on Σ_2 is given by the embedding $X = (X^i) : \Sigma_2 \to M$ and an auxilliary 1-form field on Σ_2 $\xi = (\xi_i) \in \Omega^1(\Sigma_2, X^*T^*M)$, $i \in \{1, \dots, d\}$.

Action

$$S^{(1)} = \int_{\Sigma_2} \Big(\xi_i \wedge \mathrm{d}X^i + \frac{1}{2}\,\Theta^{ij}(X)\,\xi_i \wedge \xi_j \Big)$$

This is a topological field theory on $C^\infty(T\Sigma_2, T^*M)$.

(Cattaneo & Felder, math.QA/0102108)

Courant σ-model

- Courant algebroid (= symplectic Lie 2-algebroid).
- A 3d **membrane** worldvolume Σ_3.
- $\alpha = (\alpha^I) \in \Omega^1(\Sigma_3, X^*E)$ and $\phi = (\phi_i) \in \Omega^2(\Sigma_3, X^*T^*M)$.
- Choose a local basis of sections $\{\psi_I\}$ of $E \to M$ s..t. the fibre metric $h_{IJ} := \langle \psi_I, \psi_J \rangle$ is constant, $I \in \{1, \dots, 2d\}$.
- Def. the anchor matrix $\rho(\psi_I) = P_I{}^i(x)\, \partial_i$, and the 3-form $T_{IJK}(x) := [\psi_I, \psi_J, \psi_K]_E$.

Action

$$S^{(2)} = \int_{\Sigma_3} \Big(\phi_i \wedge \mathrm{d}X^i \; + \; \frac{1}{2} h_{IJ}\, \alpha^I \wedge \mathrm{d}\alpha^J - P_I{}^i(X)\, \phi_i \wedge \alpha^I + \\ + \; \frac{1}{6} T_{IJK}(X)\, \alpha^I \wedge \alpha^J \wedge \alpha^K \Big)$$

H-space

- Standard Courant algebroid $C = TM \oplus T^*M$ twisted by $H = \frac{1}{6}\, H_{ijk}(x)\, \mathrm{d}x^i \wedge \mathrm{d}x^j \wedge \mathrm{d}x^k$.
- Structure maps:
  - Antisymmetrized H-twisted Courant-Dorfman bracket.
  - The usual pairing between TM and T^*M and ρ trivial
- Assume $TM \cong M \times \mathbb{R}^d$ to keep only H-flux.
- Write $(\alpha^I) := (\alpha^1, \ldots, \alpha^d, \xi_1, \ldots, \xi_d)$ and integrate out ϕ_i.

H-twisted Poisson σ-model

$$\begin{aligned} \widetilde{S}^{(1)} &= \oint_{\Sigma_2} \Big(\xi_i \wedge \mathrm{d}X^i + \frac{1}{2}\, \Theta^{ij}(X)\, \xi_i \wedge \xi_j \Big) \\ &+ \int_{\Sigma_3} \frac{1}{6}\, H_{ijk}(X)\, \mathrm{d}X^i \wedge \mathrm{d}X^j \wedge \mathrm{d}X^k \end{aligned}$$

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R-space

- R-twisted C-D bracket; all other structure maps don't change.

Non-topological quasi-Poisson σ-model, constant R-flux

$$S_R^{(2)} = \oint_{\Sigma_2} \Big(\eta_I \wedge \mathrm{d}X^I + \frac{1}{2}\Theta^{IJ}(X)\,\eta_I \wedge \eta_J \Big) + \oint_{\Sigma_2} \frac{1}{2} G^{IJ}\,\eta_I \wedge *\eta_J$$

$$\Theta = (\Theta^{IJ}) = \begin{pmatrix} R^{ijk}\,p_k & \delta^i{}_j \\ -\delta_i{}^j & 0 \end{pmatrix} \quad \text{and} \quad (G^{IJ}) = \begin{pmatrix} g^{ij} & 0 \\ 0 & 0 \end{pmatrix}$$

- $[\Theta,\Theta]_S = \Lambda^3\Theta^\sharp(H)$, where $H = \frac{1}{6} R^{ijk}\,\mathrm{d}p_i \wedge \mathrm{d}p_j \wedge \mathrm{d}p_k$.
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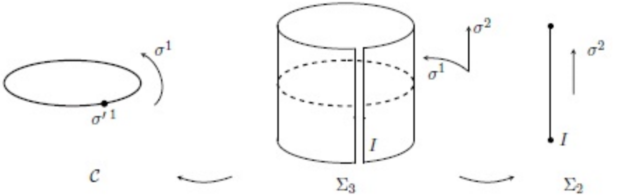
Boundary conditions

- Consider $M = \mathbb{T}^2 \times S^1$. Elliptic monodromies induce twisted BCs on closed str. Canonical quantization results to NC fibres:
$$\{x^i, x^j\}_Q = Q^{ij}{}_k \, \tilde{p}^{\,k} \quad \text{and} \quad \{x^i, \tilde{p}^{\,j}\}_Q = 0 = \{\tilde{p}^{\,i}, \tilde{p}^{\,j}\}_Q ,$$
with constant Q-flux. **Winding number dependance**.
- Closed str. on an orbifold $\rightarrow$ Open str. on the covering space.
- In CFT, monodromy=twist field at some pt $\Rightarrow$ branch pt.
- Stokes' theorem: $\Sigma_3 \rightarrow \Sigma_2 \;\Rightarrow$ branch cut I.



Motivation

- Choose BCs on I appropriate for an open str. twisted Poisson σ-model.
- Take the topological limit $g \ll R$.
- The propagator is

$$\big\langle X^I(w)\,\eta_J(z)\big\rangle = \frac{\mathrm{i}\,\hbar}{2\pi}\,\delta^I{}_J\,\mathrm{d}_z\phi^h(z,w)\ ,$$

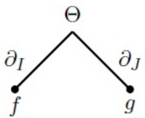
where $\mathrm{d}_z := \mathrm{d}z\,\frac{\partial}{\partial z} + \mathrm{d}\overline{z}\,\frac{\partial}{\partial \overline{z}}$ and $\phi^h(z,w)$ is the **harmonic angle**.

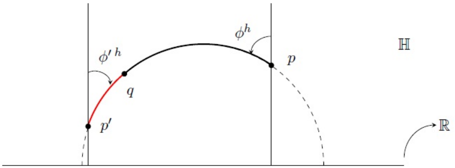
The Feynman diagram expansion reproduces Kontsevich's graphical expansion for global deformation quantization of a twisted Poisson structure.
(Cattaneo & Felder, math.QA/9902090, hep-th/0111028)

Star product

- Formality map = a sequence of L_∞ morphisms U_n.

$$f \star g := \sum_{n=0}^{\infty} \frac{(\mathrm{i}\,\hbar)^n}{n!} U_n(\Theta, \dots, \Theta)(f, g)$$





- Jacobiator quantization, i.e. $\Pi = [\Theta, \Theta]_S$.

$$[f, g, h]_\star := \sum_{n=0}^{\infty} \frac{(\mathrm{i}\,\hbar)^n}{n!} U_{n+1}(\Pi, \Theta, \dots, \Theta)(f, g, h) \equiv \Phi(\Pi)(f, g, h)$$

Derivation properties

- $\exists$ a **formality condition** s.t. U_n is an L_∞ morphism.
- Associator.

$$[f,g,h]_\star = \frac{2\,\mathrm{i}}{\hbar}\big((f\star g)\star h - f\star(g\star h)\big)$$

- Derivation properties.

$$\begin{aligned}\mathrm{i}\,\hbar\,\Phi(\mathrm{d}_\Theta\Pi)\,(f,g,h) = \;& f\star[g,h,k]_\star - [f\star g,h,k]_\star + \\ &+[f,g\star h,k]_\star - [f,g,h\star k]_\star + [f,g,h]_\star\star k\end{aligned}$$

  where $\mathrm{d}_\Theta\Pi \;:=\; [\Pi,\Theta]_{\mathrm{S}}$.
- l.h.s. is equal to zero for constant R-flux. **"Pentagon identity"**.

Constant R-flux

- Nonassociative star product.

$$f \star g = \mu_2\Big(\exp\big(\tfrac{\mathrm{i}\,\hbar}{2}\,\Theta^{IJ}\,\partial_I \otimes \partial_J\big)(f \otimes g)\Big)$$

- Associator.

$$[f,g,h]_\star = \frac{4\,\mathrm{i}}{\hbar}\,\Big[\,\tilde{\star}\Big(\sinh\big(\tfrac{\hbar^2}{4}\,R^{ijk}\,\partial_i{\otimes}\partial_j{\otimes}\partial_k\big)(f{\otimes}g{\otimes}h)\Big)\,\Big]_{\tilde{p}\to p}$$

- The associator defines a quantization of the Nambu-Poisson structure defined by Π

$$[f,g,h]_\star = 6\,\mathrm{i}\,\hbar\,\{f,g,h\}_\Theta + \mathcal{O}\big(\hbar^2\big)\ .$$

Leibnitz rule $\Longrightarrow$ pentagon identity at the quantum level.

Seiberg-Witten maps (generalities)

YM	NCYM
1. gauge field a_μ, parameter λ	1. $\hat{A}_\mu$ and $\hat{\Lambda}$
2. $\delta_\lambda a_\mu = \partial_\mu \lambda$	2. $\delta_{\hat{\Lambda}} \hat{A}_\mu = \partial_\mu \hat{\Lambda} + \mathrm{i}\,[\hat{\Lambda}\,, \hat{A}_\mu]_\star$

$\hat{A}(a)$ and $\hat{\Lambda}(\lambda, a)$ are the Seiberg-Witten maps.

(Seiberg-Witten)

- Covariantizing map.

$$\mathcal{D} : x^\mu \longmapsto \hat{x}^\mu = x^\mu + \theta^{\mu\nu}\, \hat{A}_\nu(x)$$

- Gauge transformations.

$$\theta \mapsto \theta' = \theta\,(1 + \hbar\, f\, \theta)^{-1}\ ,$$

then

$$\mathcal{D}(f \star_{\theta'} g) = \mathcal{D}f \star_\theta \mathcal{D}g$$

(Jurčo, Schupp & Wess)

Seiberg-Witten maps (on gerbes)

- On gerbes:
  - $H = \mathrm{d}B_\alpha$ on each patch,
  - $F_{\alpha\beta} := B_\beta - B_\alpha = \mathrm{d}a_{\alpha\beta}$ on overlaps,
  - $\lambda_{\alpha\beta\gamma}$ on triple overlaps.
- Θ can be locally untwisted by B_α to:
$$\Theta_\alpha = \Theta\,(1 - \hbar\, B_\alpha\, \Theta)^{-1}.$$
- Θ_α (Poisson) and $\star_\alpha$ (associative) are related by covariantizing maps computed from $a_{\alpha\beta}$.

(Jurčo, Schupp & Wess)

Seiberg-Witten maps (on phase space)

- Replace patch index α by a constant momentum vector $\tilde{p}$.

$$\Theta_{\tilde{p}} = \begin{pmatrix} \hbar\, R^{ijk}\, \tilde{p}_k & \delta^i{}_j \\ -\delta_i{}^j & 0 \end{pmatrix} \quad \text{and} \quad B_{\tilde{p}} = \begin{pmatrix} 0 & 0 \\ 0 & R^{ijk}\,(p_k - \tilde{p}_k) \end{pmatrix}$$

- 1-connection is given by:

$$a_{\tilde{p},\tilde{p}\,'} = R^{ijk}\, p_i\, (\tilde{p}_k - \tilde{p}_k')\, \mathrm{d}p_j$$

1. For H fixed the $\mathcal{D}$'s are constructed from the gauge potential:
$$A = A_I(x)\, \mathrm{d}x^I = a_i(x,p)\, \mathrm{d}x^i + \tilde{a}^i(x,p)\, \mathrm{d}p_i \ .$$
2. Relationship between $\star_0$ and $\star_{\tilde{p}}$.
$$f \star g = \big[\mathcal{D}_{\tilde{p}}(f \star g)\big]_{\tilde{p}\to p} = \big[\mathcal{D}_{\tilde{p}} f \star_0 \mathcal{D}_{\tilde{p}} g\big]_{\tilde{p}\to p}$$
SW map can be computed in closed form.

Definitions

Objective

Find a suitable higher analogue of Weyl quantization.

A 2-*vector space* is a linear category $\mathscr{V} = (\mathscr{V}_0, \mathscr{V}_1)$ of

1. a vector space of objects $\mathscr{V}_0$
2. a vector space of morphisms $\mathscr{V}_1$
3. a source and a target maps $\mathsf{s}, \mathsf{t} : \mathscr{V}_1 \rightrightarrows \mathscr{V}_0$
4. an inclusion map $\mathbb{1} : \mathscr{V}_0 \to \mathscr{V}_1$, $v \mapsto \mathbb{1}_v$.

Lie 2-algebras

A Lie 2-algebra is a 2-vector space $\mathscr{V}$ together with an antisymmetric bilinear bracket $[-,-]_{\mathscr{V}} : \mathscr{V} \times \mathscr{V} \to \mathscr{V}$ and an antisymmetric trilinear Jacobiator isomorphism on objects satisfying a higher Jacobi identity.

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More definitions

L_∞- algebras

An L_∞-algebra is a graded vector space V with a collection of totally (graded) antisymmetric n-brackets $[-,\dots,-] : \bigwedge^n V \to V$ of degree $n-2$ satisfying *higher* or *homotopy Jacobi identities*.

A 2-*term* L_∞-*algebra* is one with underlying graded vector space $V = V_0 \oplus V_1$; it has vanishing n-brackets for $n > 3$.

2-term L_∞-algebras are the same things as Lie 2-algebras.

(Baez & Crans, math.QA/0307263)

Even more definitions

A *tensor* or *monoidal* category $\mathscr{C} = (\mathscr{C}_0, \mathscr{C}_1)$ has

- an exterior product $\otimes : \mathscr{C} \times \mathscr{C} \to \mathscr{C}$
- an identity object $\mathbb{1} \in \mathscr{C}_0$
- unity isomorphisms $\mathbb{1}_X := \mathbb{1} \otimes X \cong X \cong X \otimes \mathbb{1}$ in $\mathscr{C}_1$, $\forall\ X \in \mathscr{C}_0$
- associator isomorphisms $\mathscr{P}_{X,Y,Z} : (X \otimes Y) \otimes Z \xrightarrow{\approx} X \otimes (Y \otimes Z)$, $\forall\ X, Y, Z \in \mathscr{C}_0$.

For higher associators to be consistent, $\mathbb{1}_X$ and $\mathscr{P}$ must satisfy:

1. the **pentagon identities** (5 bracketings of 4 objects)
2. the triangle identities (associator is compatible with the unity)

Braiding

$\mathscr{C}$ is *braided* if $\exists\ \mathscr{B}_{X,Y} : X \otimes Y \xrightarrow{\approx} Y \otimes X$, $\forall\ X, Y \in \mathscr{C}_0$.
$\mathscr{B}$ are called **commutativity relations**.

Lie-2 groups

A 2-*group* is a monoidal category in which every object and morphism has an inverse.

A *Lie 2-group* is a pair $\mathscr{G} = (\mathscr{G}_0, \mathscr{G}_1)$ of objects in the category of smooth manifolds and smooth maps with

1. source and target maps $\mathsf{s}, \mathsf{t} : \mathscr{G}_1 \rightrightarrows \mathscr{G}_0$
2. vertical multiplication $\circ : \mathscr{G}_1 \times \mathscr{G}_1 \to \mathscr{G}_1$
3. horizontal multiplication functor $\otimes : \mathscr{G} \times \mathscr{G} \to \mathscr{G}$
4. identity object 1
5. inversion functor $(-)^{-1} : \mathscr{G} \to \mathscr{G}$
6. associator, left and right units as before
7. units and counits $g \otimes g^{-1} \cong 1 \cong g^{-1} \otimes g$

R-space and Q-space Lie 2-algebras

- Let $V \cong \mathbb{R}^{2d}$ and define $[-,-]_R : V \wedge V \to V$ by the R-space commutation relations. This is a *pre-Lie algebra*.
- $V_0 = V_1 = V$ and let $\mathrm{d} := [-] : V_1 \to V_0$ be the identity map. This is a 2-term L_∞-algebra = the R-**space Lie** 2-**algebra** $\mathscr{V}$.

> $\mathscr{V}$ can not be directly integrated to a Lie 2-group.

- Take the Q-space algebra $\mathfrak{g}$.
- 2-term L_∞-algebra: $\big(\tilde{V}_1 = \mathbb{R} \xrightarrow{\tilde{\mathrm{d}}} \tilde{V}_0 = \mathfrak{g}\big)$ with 3-cocycle:
$$j(\hat{x}^i, \hat{x}^j, \hat{x}^k) = R^{ijk} \ .$$
This is the Q-**space Lie** 2-**algebra** $\tilde{\mathscr{V}}$.

Integrating 2-groups

- $(\mathfrak{g}, \mathbb{R}, j)$ exponentiates $(G, U(1), \varphi)$, where G=d-dim Heisenberg and φ=associator.
  - To exponentiate $[j] \in H^3(\mathfrak{g}, \mathbb{R})$ to a *compact* element $[\varphi] \in H^3(G, U(1))$, we need to restrict the space of 3-cocycles to a lattice $\Lambda \cong \mathbb{Z}^d$.
  - Λ injects into Γ= cocompact lattice in G. G/Γ is a **Heisenberg nilmanifold**.
  - Λ is equipped with an inner product and a dual pairing Σ.
- $\hat{\tilde{p}}_i \to \hat{p}_i$ with braiding : $\mathscr{B}_{g,h} = \big(g\,h\,,\,\beta(g,h)\big)$.

This braided monoidal category $\mathscr{G}$ is the Lie 2-group that integrates the Lie 2-algebra $\mathscr{V}$.

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Explicit construction

- Exponentiation of the Lie 2-algebra generators.

$$\hat{Z}^a = \exp\big(2\pi\,\mathrm{i}\,(\Sigma^{-1})_i{}^a\,\hat{x}^i\big)$$
$$\hat{P}_\xi = \exp\big(\mathrm{i}\,\xi^i\,\hat{p}_i\big),$$

where $a = 1,\dots,d$ and $\xi = (\xi^i) \in \mathbb{R}^d$.

- Commutation relations.

$$\hat{Z}^a \otimes \hat{Z}^b = \hat{P}_{\xi_R^{ab}} \otimes \hat{Z}^b \otimes \hat{Z}^a$$
$$\hat{Z}^a \otimes \hat{P}_\xi = \mathrm{e}^{\,2\pi\,\mathrm{i}\,\hbar\,(\Sigma^{-1})_i{}^a\,\xi^i}\ \hat{P}_\xi \otimes \hat{Z}^a$$
$$\hat{P}_\xi \otimes \hat{P}_{\xi'} = \hat{P}_{\xi'} \otimes \hat{P}_\xi$$

where $\big(\xi_R^{ab}\big)^i = -4\pi^2\,(\Sigma^{-1})_j{}^a\,R^{ijk}\,(\Sigma^{-1})_k{}^b$.

Explicit construction

- Associativity relation.

$$\big(\hat{Z}^a \otimes \hat{Z}^b\big) \otimes \hat{Z}^c = \mathrm{e}^{-2\pi\,\mathrm{i}\,\hbar\,R^{abc}}\ \hat{Z}^a \otimes \big(\hat{Z}^b \otimes \hat{Z}^c\big)\ ,$$

  where $R^{abc} = 2\pi^2\,R^{ijk}\,(\Sigma^{-1})_i{}^a\,(\Sigma^{-1})_j{}^b\,(\Sigma^{-1})_k{}^c$.

- The tricharacter

$$\varphi(m,n,q) = \mathrm{e}^{-2\pi\,\mathrm{i}\,\hbar\,R^{abc}\,m_a\,n_b\,q_c}\ ,$$

  where $m_a \in \Lambda^* \cong \mathbb{Z}^d$, obeys the required pentagonal cocycle condition.

- Higher associativity relations in $\mathscr{G}$ were computed. $\Rightarrow$ Quantization of the fundamental identity for Nambu-Poisson structures is encoded in $\mathscr{G}$.

Convolution star product

- Consider $C^\infty(T^*M)$ on $T^*M = \mathbb{T}^d \times (\mathbb{R}^d)^*$. Embed it as an algebra object $\mathcal{A}$ of $\mathscr{G}$.
- Categorify the Weyl map: defined it as the linear isomorphism
$$\mathscr{W}\left(\mathrm{e}^{\,\mathrm{i}\,k_I\,x^I}\right) = \hat{W}(m,\xi) := \exp\left(\mathrm{i}\,k_I\,\hat{x}^I\right) ,$$
where $(k_I) = (k_1,\dots,k_d,\xi^1,\dots,\xi^d)$ with $k_i = 2\pi\,(\Sigma^{-1})_i{}^a\,m_a$ and extended by linearity (Fourier expansion).

Star product definition

$$\mathscr{W}(f \circledast g) := \mathscr{W}(f) \otimes \mathscr{W}(g)$$

- The NA $\circledast$-product satisfies the associativity relation of the category; i.e. $(\mathcal{A}\,,\circledast)$ really is an object of $\mathscr{G}$.

Summary

- Courant σ-model $\longrightarrow$ twisted Poisson σ-model that offers a *geometric interpretation* of the R-flux background.
- Deformation quantization gave us the nonassociative star product as well as quantization of Nambu-Poisson brackets.
- SW maps from NA to associative star products were computed.
- Using the appropriate categorical formalism the Weyl quantization map has been categorified and the petinent NA convolution product was computed.
- The two products coincide giving us a categorical version of Kathotia's theorem.