The homogeneity theorem for ten- and eleven-dimensional supergravities

José Figueroa-O'Farrill

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- $g, F, \ldots$ are subject to Einstein–Maxwell-like PDEs

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- Explicitly,

$$d \star F = \tfrac{1}{2} F \wedge F$$

$$\mathrm{Ric}(X, Y) = \tfrac{1}{2}\langle \iota_X F, \iota_Y F\rangle - \tfrac{1}{6} g(X, Y)|F|^2$$

together with $dF = 0$

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- It is convenient to organise this information according to how much “supersymmetry” the background preserves.

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- such spinor fields are called **Killing spinors**

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- a background is said to be ν-**BPS**, where $\nu = \frac{n}{32}$

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where the second row are now known to be homogeneous!

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- It is a very useful invariant of a supersymmetric supergravity background

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- This is the “right” working notion in supergravity

The homogeneity theorem

Empirical Fact

Every known ν-BPS background with $\nu > \frac{1}{2}$ is homogeneous.

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In fact, vector fields in the Killing superalgebra already span the tangent spaces to every point of M

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Generalisations

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Every ν-BPS background of type IIB supergravity with $\nu > \frac{1}{2}$ is homogeneous.

Every ν-BPS background of type I and heterotic supergravities with $\nu > \frac{1}{2}$ is homogeneous.

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The theorems actually prove the strong version of the conjecture: that the symmetries which are generated from the supersymmetries already act (locally) transitively.

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This actually only shows local homogeneity.

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- we have learnt **a lot** (about string theory) from supersymmetric supergravity backgrounds, so their classification could teach us even more

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$$\left\{\begin{matrix}\text{Ad}(H)\text{-invariant}\\ \text{tensors on } \mathfrak{m}\end{matrix}\right\} \leftrightarrow \left\{\begin{matrix}H\text{-invariant}\\ \text{tensors on } T_pM\end{matrix}\right\} \leftrightarrow \left\{\begin{matrix}G\text{-invariant}\\ \text{tensor fields on } M\end{matrix}\right\}$$

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subject to some algebraic equations which are given purely in terms of the structure constants of $\mathfrak{g}$ (and $\mathfrak{h}$).

Skip technical details

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$$(\delta\varphi)_{ijk} = -\tfrac{3}{2} f_{m[i}{}^{n}\varphi^{m}{}_{jk]n} - 3U_{m[i}{}^{n}\varphi^{m}{}_{jk]n} - U_{m}{}^{mn}\varphi_{nijk}$$

where $U_{ijk} = f_{i(jk)}$

Homogeneous Ricci curvature

Finally, the Ricci tensor for a homogeneous (reductive) manifold is given by

$$R_{ij} = -\tfrac{1}{2} f_i{}^{k\ell} f_{jk\ell} - \tfrac{1}{2} f_{ik}{}^{\ell} f_{j\ell}{}^{k} + \tfrac{1}{2} f_{ik}{}^{a} f_{aj}{}^{k} + \tfrac{1}{2} f_{jk}{}^{a} f_{ai}{}^{k} - \tfrac{1}{2} f_{k\ell}{}^{\ell} f^{k}{}_{ij} - \tfrac{1}{2} f_{k\ell}{}^{\ell} f^{k}{}_{ji} + \tfrac{1}{4} f_{k\ell i} f^{k\ell}{}_{j}$$

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It is now a matter of assembling these ingredients to write down the supergravity field equations in a homogeneous Ansatz.

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- For each such geometry parametrise the space of invariant lorentzian metrics $(\gamma_1, \gamma_2, \ldots)$ and invariant closed 4-forms $(\varphi_1, \varphi_2, \ldots)$
- Plug them into the supergravity field equations to get (nonlinear) algebraic equations for the γ_i, φ_i

Methodology

Classifying homogeneous supergravity backgrounds of a certain type involves now the following steps:

- Classify the desired homogeneous geometries
- For each such geometry parametrise the space of invariant lorentzian metrics $(\gamma_1, \gamma_2, \dots)$ and invariant closed 4-forms $(\varphi_1, \varphi_2, \dots)$
- Plug them into the supergravity field equations to get (nonlinear) algebraic equations for the γ_i, φ_i
- Solve the equations!

Homogeneous lorentzian manifolds I

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Definition

The action of G on M is **proper** if the map $G \times M \to M \times M$, $(\gamma, m) \mapsto (\gamma \cdot m, m)$ is proper (i.e., inverse image of compact is compact). In particular, proper actions have compact stabilisers.

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Theorem (Kowalsky, 1996)

If a simple Lie group acts transitively and non-properly on a lorentzian manifold (M, g), then (M, g) is locally isometric to (anti) de Sitter spacetime.

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This means that we need only classify Lie subalgebras corresponding to *compact* Lie subgroups!

Some recent classification results

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- Homogeneous M2-duals: $\mathfrak{g} = \mathfrak{so}(3,2) \oplus \mathfrak{so}(N)$ for $N > 4$
  JMF+Ungureanu (in preparation)

Summary and outlook

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- In particular, we can “dial up” a semisimple G and hope to solve the homogeneous supergravity equations with symmetry G
- Checking supersymmetry is an additional problem, perhaps it can be done at the same time by considering homogeneous supermanifolds

JMF+Santi+Spiro (in progress)