Black holes, the Van der Waals gas, compressibility and the speed of sound

Brian P. Dolan

Dept. of Mathematics, Heriot-Watt University, Edinburgh
and
Maxwell Institute for Mathematical Sciences, Edinburgh

EMPG, 25th Sept. 2013

Outline

- Review of black hole thermodynamics
  - Entropy and Temperature
  - 1st and 2nd laws
  - Hawking radiation
  - Smarr relation
- Pressure and Enthalpy
  - Enthalpy and the 1st law
  - Volume
- Equation of state
  - Critical behaviour
  - Compressibility
  - Speed of sound
- Conclusions

Entropy and Temperature

- Entropy: $S \propto \frac{A}{\ell_{Pl}^2}$ ($\ell_{Pl}^2 = \hbar G/c^3$, $G = c = 1$).
  Bekenstein (1972)
- Temperature, $T = \frac{\kappa\hbar}{2\pi}$: $\kappa =$surface gravity. Hawking (1974)

Schwarzschild black-hole: $\kappa = \frac{1}{4M}$

$$T = \frac{\hbar}{8\pi M}\,.$$

Solar mass black hole: $T = 6 \times 10^{-8}\, K$, $S \approx 10^{78}$.

- Internal energy $U(S)$: $T = \frac{\partial U}{\partial S}$.
  Identify $M = U(S) \quad \Rightarrow \quad dM = TdS$.
  Schwarzschild: $r_h = 2M$, $A = 4\pi r_h^2 = 16\pi M^2$,
  $dA = 32\pi M dM$, $\frac{TdA}{\ell_{Pl}^2} = 4dM$.

Entropy and Temperature

- Entropy: $S \propto \frac{A}{\ell_{Pl}^2}$ $(\ell_{Pl}^2 = \hbar G/c^3,\ G = c = 1)$.
  Bekenstein (1972)
- Temperature, $T = \frac{\kappa\hbar}{2\pi}$: $\kappa =$surface gravity. Hawking (1974)

Schwarzschild black-hole: $\kappa = \frac{1}{4M}$

$$T = \frac{\hbar}{8\pi M}\ .$$

- Solar mass black hole: $T = 6 \times 10^{-8}\ K$, $S \approx 10^{78}$.
- Internal energy $U(S)$: $T = \frac{\partial U}{\partial S}$.
  Identify $M = U(S) \quad \Rightarrow \quad dM = TdS$.
  Schwarzschild: $r_h = 2M$, $A = 4\pi r_h^2 = 16\pi M^2$,
  $dA = 32\pi M dM$, $\frac{TdA}{\ell_{Pl}^2} = 4dM$.

Entropy and Temperature

- Entropy: $S \propto \frac{A}{\ell_{Pl}^2}$ $(\ell_{Pl}^2 = \hbar G/c^3,\ G = c = 1)$.
  Bekenstein (1972)
- Temperature, $T = \frac{\kappa\hbar}{2\pi}$: $\kappa =$surface gravity. Hawking (1974)

Schwarzschild black-hole: $\kappa = \frac{1}{4M}$

$$T = \frac{\hbar}{8\pi M}\,.$$

Solar mass black hole: $T = 6 \times 10^{-8}\ K$, $S \approx 10^{78}$.

Internal energy $U(S)$: $T = \frac{\partial U}{\partial S}$.
Identify $M = U(S) \quad \Rightarrow \quad dM = TdS$.
Schwarzschild: $r_h = 2M$, $A = 4\pi r_h^2 = 16\pi M^2$,
$dA = 32\pi M dM$, $\frac{TdA}{\ell_{Pl}^2} = 4dM$.

Entropy and Temperature

- Entropy: $S \propto \frac{A}{\ell_{Pl}^2}$ $(\ell_{Pl}^2 = \hbar G/c^3,\ G = c = 1)$.
  Bekenstein (1972)
- Temperature, $T = \frac{\kappa\hbar}{2\pi}$: $\kappa =$surface gravity. Hawking (1974)

Schwarzschild black-hole: $\kappa = \frac{1}{4M}$

$$T = \frac{\hbar}{8\pi M}\,.$$

Solar mass black hole: $T = 6 \times 10^{-8}\,K$, $S \approx 10^{78}$.

Internal energy $U(S)$: $T = \frac{\partial U}{\partial S}$.

Identify $M = U(S) \quad \Rightarrow \quad dM = TdS$.

Schwarzschild: $r_h = 2M$, $A = 4\pi r_h^2 = 16\pi M^2$,

$dA = 32\pi M dM$, $\frac{TdA}{\ell_{Pl}^2} = 4dM$.

Entropy and Temperature

- Entropy: $S \propto \frac{A}{\ell_{Pl}^2}$ $(\ell_{Pl}^2 = \hbar G/c^3,\ G = c = 1)$.
  Bekenstein (1972)
- Temperature, $T = \frac{\kappa \hbar}{2\pi}$: $\kappa =$surface gravity. Hawking (1974)

Schwarzschild black-hole: $\kappa = \frac{1}{4M}$

$$T = \frac{\hbar}{8\pi M}\,.$$

Solar mass black hole: $T = 6 \times 10^{-8}\,K$, $S \approx 10^{78}$.

- Internal energy $U(S)$: $T = \frac{\partial U}{\partial S}$.
  Identify $M = U(S) \quad \Rightarrow \quad dM = TdS$.
  Schwarzschild: $r_h = 2M$, $A = 4\pi r_h^2 = 16\pi M^2$,
  $dA = 32\pi M dM$, $\frac{TdA}{\ell_{Pl}^2} = 4dM$.

Entropy and Temperature

- Entropy: $S \propto \frac{A}{\ell_{Pl}^2}$ $(\ell_{Pl}^2 = \hbar G/c^3,\ G = c = 1)$.
  Bekenstein (1972)
- Temperature, $T = \frac{\kappa\hbar}{2\pi}$: $\kappa =$surface gravity. Hawking (1974)

Schwarzschild black-hole: $\kappa = \frac{1}{4M}$

$$T = \frac{\hbar}{8\pi M}\,.$$

Solar mass black hole: $T = 6 \times 10^{-8}\,K$, $S \approx 10^{78}$.

- Internal energy $U(S)$: $T = \frac{\partial U}{\partial S}$.
  Identify $M = U(S) \quad \Rightarrow \quad dM = TdS$.
  Schwarzschild: $r_h = 2M$, $A = 4\pi r_h^2 = 16\pi M^2$,
  $dA = 32\pi M dM$, $\frac{TdA}{\ell_{Pl}^2} = 4dM$.

Entropy and Temperature

- Entropy: $S \propto \frac{A}{\ell_{Pl}^2}$ $(\ell_{Pl}^2 = \hbar G/c^3,\ G = c = 1)$.
  Bekenstein (1972)
- Temperature, $T = \frac{\kappa\hbar}{2\pi}$: $\kappa =$surface gravity. Hawking (1974)

Schwarzschild black-hole: $\kappa = \frac{1}{4M}$

$$T = \frac{\hbar}{8\pi M}\,.$$

Solar mass black hole: $T = 6 \times 10^{-8}\ K$, $S \approx 10^{78}$.

- Internal energy $U(S)$: $T = \frac{\partial U}{\partial S}$.
  Identify $M = U(S) \quad \Rightarrow \quad dM = TdS$.
  Schwarzschild: $r_h = 2M$, $A = 4\pi r_h^2 = 16\pi M^2$,
  $dA = 32\pi M dM$, $\frac{TdA}{\ell_{Pl}^2} = 4dM$.

Entropy and Temperature

- Entropy: $S = \frac{1}{4}\frac{A}{\ell_{Pl}^2}$ $(\ell_{Pl}^2 = \hbar G/c^3,\ G = c = 1)$.
  Bekenstein (1972)
- Temperature, $T = \frac{\kappa\hbar}{2\pi}$: $\kappa =$surface gravity. Hawking (1974)

Schwarzschild black-hole: $\kappa = \frac{1}{4M}$

$$T = \frac{\hbar}{8\pi M}\,.$$

Solar mass black hole: $T = 6 \times 10^{-8}\ K$, $S \approx 10^{78}$.

- Internal energy $U(S)$: $T = \frac{\partial U}{\partial S}$.
  Identify $M = U(S) \quad \Rightarrow \quad dM = TdS$.
  Schwarzschild: $r_h = 2M$, $A = 4\pi r_h^2 = 16\pi M^2$,
  $dA = 32\pi M dM$, $\frac{TdA}{\ell_{Pl}^2} = 4dM$.

First and Second laws

- More generally $M = U(S, J, Q)$,
  (angular momentum, electric charge),

First Law of Black Hole Thermodynamics

$$dM = T\,dS + \Omega dJ + \Phi dQ.$$

- Can extract useful work in a Penrose process.
  - Kerr black hole ($Q = 0$):

$$T = \frac{1}{8\pi M}\left(1 - 4\pi^2 \frac{J^2}{S^2}\right).$$

  - Extremal: $J_{max} = \frac{S}{2\pi} \;\Rightarrow\; T = 0$.
  - Reduce $J \Rightarrow$ extract energy.
  - Maximum efficiency for fixed S: $\eta = 1 - \frac{1}{\sqrt{2}} \approx 29\%$.

Second Law of Black Hole Thermodynamics

$$\Delta S \geq 0.$$

First and Second laws

- More generally $M = U(S, J, Q)$,
  (angular momentum, electric charge),

First Law of Black Hole Thermodynamics

$$dM = T\,dS + \Omega dJ + \Phi dQ.$$

- Can extract useful work in a Penrose process.
  - Kerr black hole ($Q = 0$):

$$T = \frac{1}{8\pi M}\left(1 - 4\pi^2 \frac{J^2}{S^2}\right).$$

  - Extremal: $J_{max} = \frac{S}{2\pi} \Rightarrow T = 0$.
  - Reduce $J \Rightarrow$ extract energy.
  - Maximum efficiency for fixed S: $\eta = 1 - \frac{1}{\sqrt{2}} \approx 29\%$.

Second Law of Black Hole Thermodynamics

$$\Delta S \geq 0.$$

First and Second laws

- More generally $M = U(S, J, Q)$,
  (angular momentum, electric charge),

First Law of Black Hole Thermodynamics

$$dM = T\,dS + \Omega dJ + \Phi dQ.$$

- Can extract useful work in a Penrose process.
  - Kerr black hole ($Q = 0$):

$$T = \frac{1}{8\pi M}\left(1 - 4\pi^2 \frac{J^2}{S^2}\right).$$

  - Extremal: $J_{max} = \frac{S}{2\pi} \Rightarrow T = 0$.
  - Reduce $J \Rightarrow$ extract energy.
  - Maximum efficiency for fixed S: $\eta = 1 - \frac{1}{\sqrt{2}} \approx 29\%$.

Second Law of Black Hole Thermodynamics

$$\Delta S \geq 0.$$

First and Second laws

- More generally $M = U(S, J, Q)$,
  (angular momentum, electric charge),

First Law of Black Hole Thermodynamics

$$dM = T\,dS + \Omega dJ + \Phi dQ.$$

- Can extract useful work in a Penrose process.
  - Kerr black hole $(Q = 0)$:

$$T = \frac{1}{8\pi M}\left(1 - 4\pi^2\frac{J^2}{S^2}\right).$$

  - Extremal: $J_{max} = \frac{S}{2\pi} \Rightarrow T = 0$.
  - Reduce $J \Rightarrow$ extract energy.
  - Maximum efficiency for fixed S: $\eta = 1 - \frac{1}{\sqrt{2}} \approx 29\%$.

Second Law of Black Hole Thermodynamics

$$\Delta S \geq 0.$$

First and Second laws

- More generally $M = U(S, J, Q)$,
  (angular momentum, electric charge),

First Law of Black Hole Thermodynamics

$$dM = T\,dS + \Omega dJ + \Phi dQ.$$

- Can extract useful work in a Penrose process.
  - Kerr black hole ($Q = 0$):

$$T = \frac{1}{8\pi M}\left(1 - 4\pi^2 \frac{J^2}{S^2}\right).$$

  - Extremal: $J_{max} = \frac{S}{2\pi} \;\Rightarrow\; T = 0.$
  - Reduce $J \Rightarrow$ extract energy.
  - Maximum efficiency for fixed S: $\eta = 1 - \frac{1}{\sqrt{2}} \approx 29\%$.

Second Law of Black Hole Thermodynamics

$$\Delta S \geq 0.$$

First and Second laws

- More generally $M = U(S, J, Q)$, (angular momentum, electric charge),

First Law of Black Hole Thermodynamics

$$dM = T\,dS + \Omega dJ + \Phi dQ.$$

- Can extract useful work in a Penrose process.
  - Kerr black hole ($Q = 0$):

$$T = \frac{1}{8\pi M}\left(1 - 4\pi^2\frac{J^2}{S^2}\right).$$

  - Extremal: $J_{max} = \frac{S}{2\pi} \;\Rightarrow\; T = 0.$
  - Reduce $J \Rightarrow$ extract energy.
  - Maximum efficiency for fixed S: $\eta = 1 - \frac{1}{\sqrt{2}} \approx 29\%.$

Second Law of Black Hole Thermodynamics

$$\Delta S \geq 0.$$

First and Second laws

- More generally $M = U(S, J, Q)$,
  (angular momentum, electric charge),

First Law of Black Hole Thermodynamics

$$dM = T\,dS + \Omega dJ + \Phi dQ.$$

- Can extract useful work in a Penrose process.
  - Kerr black hole $(Q = 0)$:

$$T = \frac{1}{8\pi M}\left(1 - 4\pi^2 \frac{J^2}{S^2}\right).$$

  - Extremal: $J_{max} = \frac{S}{2\pi} \Rightarrow T = 0$.
  - Reduce $J \Rightarrow$ extract energy.
  - Maximum efficiency for fixed S: $\eta = 1 - \frac{1}{\sqrt{2}} \approx 29\%$.

Second Law of Black Hole Thermodynamics

$$\Delta S \geq 0.$$

First and Second laws

- More generally $M = U(S, J, Q)$,
  (angular momentum, electric charge),

First Law of Black Hole Thermodynamics

$$dM = T\,dS + \Omega dJ + \Phi dQ.$$

- Can extract useful work in a Penrose process.
  - Kerr black hole ($Q = 0$):

$$T = \frac{1}{8\pi M}\left(1 - 4\pi^2\frac{J^2}{S^2}\right).$$

  - Extremal: $J_{max} = \frac{S}{2\pi} \;\Rightarrow\; T = 0.$
  - Reduce $J \Rightarrow$ extract energy.
  - Maximum efficiency for fixed S: $\eta = 1 - \frac{1}{\sqrt{2}} \approx 29\%$.

Second Law of Black Hole Thermodynamics

$$\Delta S \geq 0.$$

Hawking radiation

- Heat capacity: $C = \frac{\partial U}{\partial T} = -T\frac{\partial^2 F}{\partial T^2}$.
- Free energy, $F(T)$, is Legendre transform of $U(S)$

$$F = U - TS = M - \frac{\kappa A}{8\pi}.$$

- Schwarzschild:

$$F = \frac{M}{2} = \frac{\hbar}{16\pi T}.$$

- Heat capacity:

$$C = -\frac{8\pi M^2}{\hbar} = -2S < 0. \qquad \text{Negative!}$$

- Radiates with power $\mathcal{P} \sim \frac{AT^4}{\hbar^3} \sim \frac{\hbar}{M^2}$.
  Lifetime: $\tau \sim \frac{M}{\mathcal{P}} \sim \frac{M^3}{\hbar}$, $M \sim 10^{12}\ kg \Rightarrow \tau \sim 10^{10}\ years$.
- Fixing J or $Q \neq$ stabilises the black hole.

Hawking radiation

- Heat capacity: $C = \frac{\partial U}{\partial T} = -T \frac{\partial^2 F}{\partial T^2}$.
- Free energy, $F(T)$, is Legendre transform of $U(S)$

$$F = U - TS = M - \frac{\kappa A}{8\pi}.$$

- Schwarzschild:

$$F = \frac{M}{2} = \frac{\hbar}{16\pi T}.$$

- Heat capacity:

$$C = -\frac{8\pi M^2}{\hbar} = -2S < 0.$$

Negative!

- Radiates with power $\mathcal{P} \sim \frac{[illegible]}{\hbar^3} \sim \frac{[illegible]}{M^2}$.
  Lifetime: $\tau \sim \frac{M}{\mathcal{P}} \sim \frac{M^3}{\hbar}$, $M \sim 10^{12}\ kg \Rightarrow \tau \sim 10^{10}\ years$.
- Fixing J or $Q \neq$ stabilises the black hole.

Hawking radiation

- Heat capacity: $C = \frac{\partial U}{\partial T} = -T\frac{\partial^2 F}{\partial T^2}$.
- Free energy, $F(T)$, is Legendre transform of $U(S)$

$$F = U - TS = M - \frac{\kappa A}{8\pi}.$$

- Schwarzschild:

$$F = \frac{M}{2} = \frac{\hbar}{16\pi T}.$$

- Heat capacity:

$$C = -\frac{8\pi M^2}{\hbar} = -2S < 0. \qquad \text{Negative!}$$

- Radiates with power $\mathcal{P} \sim \frac{[illegible]}{\hbar^3} \sim \frac{[illegible]}{M^2}$.
  Lifetime: $\tau \sim \frac{M}{\mathcal{P}} \sim \frac{M^3}{\hbar}$, $M \sim 10^{12}\ kg \Rightarrow \tau \sim 10^{10}\ years$.
- Fixing J or $Q \neq$ stabilises the black hole.

Hawking radiation

- Heat capacity: $C = \frac{\partial U}{\partial T} = -T\frac{\partial^2 F}{\partial T^2}$.
- Free energy, $F(T)$, is Legendre transform of $U(S)$

$$F = U - TS = M - \frac{\kappa A}{8\pi}.$$

- Schwarzschild:

$$F = \frac{M}{2} = \frac{\hbar}{16\pi T}.$$

- Heat capacity:

$$C = -\frac{8\pi M^2}{\hbar} = -2S < 0.$$

Negative!

- Radiates with power $\mathcal{P} \sim$ [illegible] $\sim \frac{\hbar}{M^2}$.
  Lifetime: $\tau \sim \frac{M}{\mathcal{P}} \sim \frac{M^3}{\hbar}$, $M \sim 10^{12}\ kg \Rightarrow \tau \sim 10^{10}\ years$.
- Fixing J or $Q \neq$ stabilises the black hole.

Hawking radiation

- Heat capacity: $C = \frac{\partial U}{\partial T} = -T \frac{\partial^2 F}{\partial T^2}$.
- Free energy, $F(T)$, is Legendre transform of $U(S)$

$$F = U - TS = M - \frac{\kappa A}{8\pi}.$$

- Schwarzschild:

$$F = \frac{M}{2} = \frac{\hbar}{16\pi T}.$$

- Heat capacity:

$$C = -\frac{8\pi M^2}{\hbar} = -2S < 0.$$

Negative!

- Radiates with power $\mathcal{P} \sim \frac{AT^4}{\hbar^3} \sim \frac{\hbar}{M^2}$.

Lifetime: $\tau \sim \frac{M}{\mathcal{P}} \sim \frac{M^3}{\hbar}$, $M \sim 10^{12}$ *kg* $\Rightarrow \tau \sim 10^{10}$ *years*.

- Fixing J or $Q \neq$ stabilises the black hole.

Hawking radiation

- Heat capacity: $C = \frac{\partial U}{\partial T} = -T\frac{\partial^2 F}{\partial T^2}$.
- Free energy, $F(T)$, is Legendre transform of $U(S)$

$$F = U - TS = M - \frac{\kappa A}{8\pi}.$$

- Schwarzschild:

$$F = \frac{M}{2} = \frac{\hbar}{16\pi T}.$$

- Heat capacity:

$$C = -\frac{8\pi M^2}{\hbar} = -2S < 0. \qquad \text{Negative!}$$

- Radiates with power $\mathcal{P} \sim \frac{AT^4}{\hbar^3} \sim \frac{\hbar}{M^2}$.
  Lifetime: $\tau \sim \frac{M}{\mathcal{P}} \sim \frac{M^3}{\hbar}$, $M \sim 10^{12}\ kg \ \Rightarrow \tau \sim 10^{10}\ years$.
- Fixing J or $Q \neq$ stabilises the black hole.

Hawking radiation

- Heat capacity: $C = \frac{\partial U}{\partial T} = -T\frac{\partial^2 F}{\partial T^2}$.
- Free energy, $F(T)$, is Legendre transform of $U(S)$

$$F = U - TS = M - \frac{\kappa A}{8\pi}.$$

- Schwarzschild:

$$F = \frac{M}{2} = \frac{\hbar}{16\pi T}.$$

- Heat capacity:

$$C = -\frac{8\pi M^2}{\hbar} = -2S < 0. \qquad \text{Negative!}$$

- Radiates with power $\mathcal{P} \sim \frac{AT^4}{\hbar^3} \sim \frac{\hbar}{M^2}$.
  Lifetime: $\tau \sim \frac{M}{\mathcal{P}} \sim \frac{M^3}{\hbar}$, $M \sim 10^{12}\ kg \Rightarrow \tau \sim 10^{10}\ years$.
- Fixing J or $Q \neq$ stabilises the black hole.

Smarr relation

Smarr (1973)

- Ordinary thermodynamics: $U(S, V, n)$ ($n =$ number of moles) is a function of extensive variables.

  U is also extensive $\Rightarrow$

$$\lambda^d U(S,V,n) = U(\lambda^d S, \lambda^d V, \lambda^d n)$$

$$\Rightarrow \quad U = S\frac{\partial U}{\partial S} + V\frac{\partial U}{\partial V} + n\frac{\partial U}{\partial n}$$ Euler equation

$$\Rightarrow \quad U = ST - VP + n\mu$$ ($\mu =$ chemical potential)

$$\Rightarrow \quad G = U + VP - ST = n\mu\,.$$ Gibbs-Duhem relation

- Black hole in D dimensions, angular momenta J_i: ($Q = 0$)

  $S \to \lambda^{D-2} S$, $J_i \to \lambda^{D-2} J_i$, $M \to \lambda^{D-3} M \quad \Rightarrow$

$$\lambda^{D-3} M(S, J_i) = M(\lambda^{D-2} S, \lambda^{D-2} J_i)$$

$$(D-3)M = (D-2)S\frac{\partial M}{\partial S} + (D-2)J_i\frac{\partial M}{\partial J_i}$$

$$\Rightarrow \quad (D-3)M = (D-2)ST + (D-2)J_i\Omega_i\,.$$

Smarr relation

Smarr (1973)

- Ordinary thermodynamics: $U(S,V,n)$ ($n =$ number of moles) is a function of extensive variables.
  U is also extensive $\Rightarrow$

$$\lambda^d U(S,V,n) = U(\lambda^d S, \lambda^d V, \lambda^d n)$$

$$\Rightarrow \quad U = S\frac{\partial U}{\partial S} + V\frac{\partial U}{\partial V} + n\frac{\partial U}{\partial n} \qquad \text{Euler equation}$$

$$\Rightarrow \quad U = ST - VP + n\mu \qquad (\mu = \text{chemical potential})$$

$$\Rightarrow \quad G = U + VP - ST = n\mu\,. \qquad \text{Gibbs-Duhem relation}$$

- Black hole in D dimensions, angular momenta J_i: $(Q = 0)$
  $S \to \lambda^{D-2} S$, $J_i \to \lambda^{D-2} J_i$, $M \to \lambda^{D-3} M \quad \Rightarrow$

$$\lambda^{D-3} M(S, J_i) = M(\lambda^{D-2} S, \lambda^{D-2} J_i)$$

$$(D-3)M = (D-2)S\frac{\partial M}{\partial S} + (D-2)J_i\frac{\partial M}{\partial J_i}$$

$$\Rightarrow \quad (D-3)M = (D-2)ST + (D-2)J_i\Omega_i\,.$$

Smarr relation

Smarr (1973)

- Ordinary thermodynamics: $U(S, V, n)$ ($n =$ number of moles) is a function of extensive variables.
  U is also extensive $\Rightarrow$

$$\begin{aligned} \lambda^d U(S,V,n) &= U(\lambda^d S, \lambda^d V, \lambda^d n) \\ \Rightarrow \quad U &= S\frac{\partial U}{\partial S} + V\frac{\partial U}{\partial V} + n\frac{\partial U}{\partial n} \qquad \text{Euler equation} \\ \Rightarrow \quad U &= ST - VP + n\mu \qquad (\mu = \text{chemical potential}) \\ \Rightarrow \quad G &= U + VP - ST = n\mu\,. \qquad \text{Gibbs-Duhem relation} \end{aligned}$$

- Black hole in D dimensions, angular momenta J_i: ($Q = 0$)
  $S \to \lambda^{D-2} S$, $J_i \to \lambda^{D-2} J_i$, $M \to \lambda^{D-3} M \quad \Rightarrow$

$$\begin{aligned} \lambda^{D-3} M(S, J_i) &= M(\lambda^{D-2} S, \lambda^{D-2} J_i) \\ (D-3)M &= (D-2)S\frac{\partial M}{\partial S} + (D-2)J_i\frac{\partial M}{\partial J_i} \\ \Rightarrow \quad (D-3)M &= (D-2)ST + (D-2)J_i\,\Omega_i\,. \end{aligned}$$

Smarr relation

Smarr (1973)

- Ordinary thermodynamics: $U(S,V,n)$ ($n=$ number of moles) is a function of extensive variables.
  U is also extensive $\Rightarrow$

$$\begin{aligned}
\lambda^d U(S,V,n) &= U(\lambda^d S, \lambda^d V, \lambda^d n) \\
\Rightarrow \qquad U &= S\frac{\partial U}{\partial S} + V\frac{\partial U}{\partial V} + n\frac{\partial U}{\partial n} && \text{Euler equation} \\
\Rightarrow \qquad U &= ST - VP + n\mu && (\mu = \text{chemical potential}) \\
\Rightarrow \qquad G &= U + VP - ST = n\mu\,. && \text{Gibbs-Duhem relation}
\end{aligned}$$

- Black hole in D dimensions, angular momenta J_i: ($Q=0$)
  $S \to \lambda^{D-2} S$, $J_i \to \lambda^{D-2} J_i$, $M \to \lambda^{D-3} M$ $\Rightarrow$

$$\begin{aligned}
\lambda^{D-3} M(S, J_i) &= M(\lambda^{D-2} S, \lambda^{D-2} J_i) \\
(D-3)M &= (D-2)S\frac{\partial M}{\partial S} + (D-2)J_i\frac{\partial M}{\partial J_i} \\
\Rightarrow \qquad (D-3)M &= (D-2)ST + (D-2)J_i\,\Omega_i\,.
\end{aligned}$$

Smarr relation

Smarr (1973)

- Ordinary thermodynamics: $U(S,V,n)$ ($n =$ number of moles) is a function of extensive variables.
  U is also extensive $\Rightarrow$

$$\begin{aligned}
\lambda^d U(S,V,n) &= U(\lambda^d S, \lambda^d V, \lambda^d n) \\
\Rightarrow \quad U &= S\frac{\partial U}{\partial S} + V\frac{\partial U}{\partial V} + n\frac{\partial U}{\partial n} && \text{Euler equation} \\
\Rightarrow \quad U &= ST - VP + n\mu && (\mu = \text{chemical potential}) \\
\Rightarrow \quad G &= U + VP - ST = n\mu\,. && \text{Gibbs-Duhem relation}
\end{aligned}$$

- Black hole in D dimensions, angular momenta J_i: ($Q = 0$)
  $S \to \lambda^{D-2} S$, $J_i \to \lambda^{D-2} J_i$, $M \to \lambda^{D-3} M$ $\Rightarrow$

$$\begin{aligned}
\lambda^{D-3} M(S, J_i) &= M(\lambda^{D-2} S, \lambda^{D-2} J_i) \\
(D-3)M &= (D-2)S\frac{\partial M}{\partial S} + (D-2)J_i\frac{\partial M}{\partial J_i} \\
\Rightarrow \quad (D-3)M &= (D-2)ST + (D-2)J_i\Omega_i\,.
\end{aligned}$$

Smarr relation

Smarr (1973)

- Ordinary thermodynamics: $U(S, V, n)$ ($n =$ number of moles) is a function of extensive variables.
  U is also extensive $\Rightarrow$

$$
\begin{aligned}
\lambda^d U(S, V, n) &= U(\lambda^d S, \lambda^d V, \lambda^d n) \\
\Rightarrow \quad U &= S\frac{\partial U}{\partial S} + V\frac{\partial U}{\partial V} + n\frac{\partial U}{\partial n} && \text{Euler equation} \\
\Rightarrow \quad U &= ST - VP + n\mu && (\mu = \text{chemical potential}) \\
\Rightarrow \quad G &= U + VP - ST = n\mu\,. && \text{Gibbs-Duhem relation}
\end{aligned}
$$

- Black hole in D dimensions, angular momenta J_i: $(Q = 0)$
  $S \to \lambda^{D-2} S$, $J_i \to \lambda^{D-2} J_i$, $M \to \lambda^{D-3} M \quad \Rightarrow$

$$
\begin{aligned}
\lambda^{D-3} M(S, J_i) &= M(\lambda^{D-2} S, \lambda^{D-2} J_i) \\
(D-3)M &= (D-2)S\frac{\partial M}{\partial S} + (D-2)J_i\frac{\partial M}{\partial J_i} \\
\Rightarrow \quad (D-3)M &= (D-2)ST + (D-2)J_i\,\Omega_i\,.
\end{aligned}
$$

Smarr relation

Smarr (1973)

- Ordinary thermodynamics: $U(S, V, n)$ ($n =$ number of moles) is a function of extensive variables.
  U is also extensive $\Rightarrow$

$$\begin{aligned} \lambda^d U(S, V, n) &= U(\lambda^d S, \lambda^d V, \lambda^d n) \\ \Rightarrow \quad U &= S\frac{\partial U}{\partial S} + V\frac{\partial U}{\partial V} + n\frac{\partial U}{\partial n} \qquad \text{Euler equation} \\ \Rightarrow \quad U &= ST - VP + n\mu \qquad (\mu = \text{chemical potential}) \\ \Rightarrow \quad G &= U + VP - ST = n\mu\,. \qquad \text{Gibbs-Duhem relation} \end{aligned}$$

- Black hole in D dimensions, angular momenta J_i: $(Q = 0)$
  $S \to \lambda^{D-2} S$, $J_i \to \lambda^{D-2} J_i$, $M \to \lambda^{D-3} M \quad \Rightarrow$

$$\begin{aligned} \lambda^{D-3} M(S, J_i) &= M(\lambda^{D-2} S, \lambda^{D-2} J_i) \\ (D-3)M &= (D-2)S\frac{\partial M}{\partial S} + (D-2)J_i\frac{\partial M}{\partial J_i} \\ \Rightarrow \quad (D-3)M &= (D-2)ST + (D-2)J_i\Omega_i\,. \end{aligned}$$

Smarr relation

Smarr (1973)

- Ordinary thermodynamics: $U(S,V,n)$ ($n =$ number of moles) is a function of extensive variables.
  U is also extensive $\Rightarrow$

$$
\begin{aligned}
\lambda^d U(S,V,n) &= U(\lambda^d S, \lambda^d V, \lambda^d n) \\
\Rightarrow \quad U &= S\frac{\partial U}{\partial S} + V\frac{\partial U}{\partial V} + n\frac{\partial U}{\partial n} && \text{Euler equation} \\
\Rightarrow \quad U &= ST - VP + n\mu && (\mu = \text{chemical potential}) \\
\Rightarrow \quad G &= U + VP - ST = n\mu\,. && \text{Gibbs-Duhem relation}
\end{aligned}
$$

- Black hole in D dimensions, angular momenta J_i: $(Q = 0)$
  $S \to \lambda^{D-2} S$, $J_i \to \lambda^{D-2} J_i$, $M \to \lambda^{D-3} M \quad \Rightarrow$

$$
\begin{aligned}
\lambda^{D-3} M(S, J_i) &= M(\lambda^{D-2} S, \lambda^{D-2} J_i) \\
(D-3)M &= (D-2)S\frac{\partial M}{\partial S} + (D-2)J_i\frac{\partial M}{\partial J_i} \\
\Rightarrow \quad (D-3)M &= (D-2)ST + (D-2)J_i\,\Omega_i\,.
\end{aligned}
$$

Smarr relation

- Doesn't work when there's a Cosmological Constant!
- Solution: (Kastor, Ray+Traschen [0904.2765])
  include Λ as a thermodynamic variable, $\Lambda \to \lambda^{-2}\Lambda$,
  (Henneaux+Teitelboim (1984)).

$$\begin{aligned} \lambda^{D-3} M(S,\Lambda,J_i) &= M(\lambda^{D-2}S,\lambda^{-2}\Lambda,\lambda^{D-2}J_i) \\ (D-3)M &= (D-2)S\frac{\partial M}{\partial S} - 2\Lambda\frac{\partial M}{\partial \Lambda} + (D-2)J_i\frac{\partial M}{\partial J_i} \\ \Rightarrow \qquad (D-3)M &= (D-2)ST - 2\Lambda\Theta + (D-2)J_i\,\Omega_i \end{aligned}$$

$\Theta := \left.\frac{\partial M}{\partial \Lambda}\right|_{S,J}$.

Smarr relation

- Doesn't work when there's a Cosmological Constant!
- Solution: (Kastor, Ray+Traschen [0904.2765])
  include Λ as a thermodynamic variable, $\Lambda \to \lambda^{-2}\Lambda$,
  (Henneaux+Teitelboim (1984)).

$$
\begin{aligned}
\lambda^{D-3} M(S,\Lambda,J_i) &= M(\lambda^{D-2}S, \lambda^{-2}\Lambda, \lambda^{D-2}J_i) \\
(D-3)M &= (D-2)S\frac{\partial M}{\partial S} - 2\Lambda\frac{\partial M}{\partial \Lambda} + (D-2)J_i\frac{\partial M}{\partial J_i} \\
\Rightarrow \quad (D-3)M &= (D-2)ST - 2\Lambda\Theta + (D-2)J_i\,\Omega_i
\end{aligned}
$$

$$\Theta := \left.\frac{\partial M}{\partial \Lambda}\right|_{S,J}.$$

Smarr relation

- Doesn't work when there's a Cosmological Constant!
- Solution: (Kastor, Ray+Traschen [0904.2765])
  include Λ as a thermodynamic variable, $\Lambda \to \lambda^{-2}\Lambda$,
  (Henneaux+Teitelboim (1984)).

$$\begin{aligned}
\lambda^{D-3} M(S,\Lambda,J_i) &= M(\lambda^{D-2}S, \lambda^{-2}\Lambda, \lambda^{D-2}J_i) \\
(D-3)M &= (D-2)S\frac{\partial M}{\partial S} - 2\Lambda\frac{\partial M}{\partial \Lambda} + (D-2)J_i\frac{\partial M}{\partial J_i} \\
\Rightarrow \quad (D-3)M &= (D-2)ST - 2\Lambda\Theta + (D-2)J_i\,\Omega_i
\end{aligned}$$

$$\Theta := \left.\frac{\partial M}{\partial \Lambda}\right|_{S,J_i}$$

Smarr relation

- Doesn't work when there's a Cosmological Constant!
- Solution: (Kastor, Ray+Traschen [0904.2765])
  include Λ as a thermodynamic variable, $\Lambda \to \lambda^{-2}\Lambda$,
  (Henneaux+Teitelboim (1984)).

$$
\begin{aligned}
\lambda^{D-3} M(S,\Lambda,J_i) &= M(\lambda^{D-2}S, \lambda^{-2}\Lambda, \lambda^{D-2}J_i) \\
(D-3)M &= (D-2)S\frac{\partial M}{\partial S} - 2\Lambda\frac{\partial M}{\partial \Lambda} + (D-2)J_i\frac{\partial M}{\partial J_i} \\
\Rightarrow \quad (D-3)M &= (D-2)ST - 2\Lambda\Theta + (D-2)J_i\,\Omega_i
\end{aligned}
$$

$$\Theta := \left.\frac{\partial M}{\partial \Lambda}\right|_{S,J_i}.$$

Smarr relation

- Doesn't work when there's a Cosmological Constant!
- Solution: (Kastor, Ray+Traschen [0904.2765])
  include Λ as a thermodynamic variable, $\Lambda \to \lambda^{-2}\Lambda$,
  (Henneaux+Teitelboim (1984)).

$$
\begin{aligned}
\lambda^{D-3} M(S,\Lambda,J_i) &= M(\lambda^{D-2}S, \lambda^{-2}\Lambda, \lambda^{D-2}J_i) \\
(D-3)M &= (D-2)S\frac{\partial M}{\partial S} - 2\Lambda\frac{\partial M}{\partial \Lambda} + (D-2)J_i\frac{\partial M}{\partial J_i} \\
\Rightarrow \qquad (D-3)M &= (D-2)ST - 2\Lambda\Theta + (D-2)J_i\,\Omega_i
\end{aligned}
$$

$$
\Theta := \left.\frac{\partial M}{\partial \Lambda}\right|_{S,J}.
$$

Enthalpy

- Pressure $P = -\frac{\Lambda}{8\pi} \Rightarrow$ mass is the enthalpy:

Enthalpy

$$M = H(S, P, J)$$

(Kastor, Ray+Traschen [0904.2765])

- Internal energy $U(S, V, J)$ is the Legendre transform:

$$U = H - PV$$

where $V := \frac{\partial M}{\partial P}$ is a thermodynamic volume.

First Law of thermodynamics

$$dU = T\,dS \;-P\,dV + \Omega dJ$$

- For $\Lambda < 0$, Penrose process can be up to 51.8% efficient (29% for $\Lambda = 0$). (BPD [1008.5023])

Enthalpy

- Pressure $P = -\frac{\Lambda}{8\pi} \Rightarrow$ mass is the enthalpy:

Enthalpy

$$M = H(S, P, J)$$

(Kastor, Ray+Traschen [0904.2765])

- Internal energy $U(S, V, J)$ is the Legendre transform:

$$U = H - PV$$

where $V := \frac{\partial M}{\partial P}$ is a thermodynamic volume.

First Law of thermodynamics

$$dU = T\,dS \; - P\,dV + \Omega dJ$$

- For $\Lambda < 0$, Penrose process can be up to 51.8% efficient (29% for $\Lambda = 0$). (BPD [1008.5023])

Enthalpy

- Pressure $P = -\frac{\Lambda}{8\pi} \Rightarrow$ mass is the enthalpy:

Enthalpy

$$M = H(S, P, J)$$

(Kastor, Ray+Traschen [0904.2765])

- Internal energy $U(S, V, J)$ is the Legendre transform:

$$U = H - PV$$

where $V := \frac{\partial M}{\partial P}$ is a thermodynamic volume.

First Law of thermodynamics

$$dU = T\,dS \; - P\,dV + \Omega dJ$$

- For $\Lambda < 0$, Penrose process can be up to 51.8% efficient (29% for $\Lambda = 0$). (BPD [1008.5023])

Enthalpy

- Pressure $P = -\frac{\Lambda}{8\pi} \Rightarrow$ mass is the enthalpy:

Enthalpy

$$M = H(S, P, J)$$

(Kastor, Ray+Traschen [0904.2765])

- Internal energy $U(S, V, J)$ is the Legendre transform:

$$U = H - PV$$

where $V := \frac{\partial M}{\partial P}$ is a thermodynamic volume.

First Law of thermodynamics

$$dU = T\,dS \; - P\,dV + \Omega dJ$$

- For $\Lambda < 0$, Penrose process can be up to 51.8% efficient (29% for $\Lambda = 0$). (BPD [1008.5023])

Enthalpy

- Pressure $P = -\frac{\Lambda}{8\pi} \Rightarrow$ mass is the enthalpy:

Enthalpy

$$M = H(S, P, J)$$

(Kastor, Ray+Traschen [0904.2765])

- Internal energy $U(S, V, J)$ is the Legendre transform:

$$U = H - PV$$

where $V := \frac{\partial M}{\partial P}$ is a thermodynamic volume.

First Law of thermodynamics

$$dU = T\,dS \; - P\,dV + \Omega dJ$$

- For $\Lambda < 0$, Penrose process can be up to 51.8% efficient (29% for $\Lambda = 0$). (BPD [1008.5023])

Thermodynamic Volume

- In $4-D$ the thermodynamic volume evaluates to

$$V = \frac{1}{3}\left(r_h \mathcal{A}_h + \frac{4\pi J^2}{M} \right),$$

where $\mathcal{A}_h$ is area of the event horizon and r_h the radius (Cvetič *et al.* [1012.2888], BPD [1106.6260]).

- Reverse Isometric inequality,
- As $J \to 0$, $\mathcal{A}_h \to 4\pi r_h^2$ and

$$V \to V_0 := \frac{4\pi}{3} r_h^3 = \frac{4}{3}\left(\frac{S^3}{\pi}\right)^{1/2}.$$

As $J \to 0$, V and S are not independent.

Thermodynamic Volume

- In $4-D$ the thermodynamic volume evaluates to

$$V = \frac{1}{3}\left(r_h \mathcal{A}_h + \frac{4\pi J^2}{M} \right),$$

where $\mathcal{A}_h$ is area of the event horizon and r_h the radius (Cvetič *et al.* [1012.2888], BPD [1106.6260]).

- Reverse Isometric inequality,
- As $J \to 0$, $\mathcal{A}_h \to 4\pi r_h^2$ and

$$V \to V_0 := \frac{4\pi}{3} r_h^3 = \frac{4}{3}\left(\frac{S^3}{\pi}\right)^{1/2}.$$

As $J \to 0$, V and S are not independent.

Thermodynamic Volume

- In $4-D$ the thermodynamic volume evaluates to

$$V = \frac{1}{3}\left(r_h \mathcal{A}_h + \frac{4\pi J^2}{M} \right),$$

where $\mathcal{A}_h$ is area of the event horizon and r_h the radius (Cvetič *et al.* [1012.2888], BPD [1106.6260]).

- Reverse Isometric inequality,
- As $J \to 0$, $\mathcal{A}_h \to 4\pi r_h^2$ and

$$V \to V_0 := \frac{4\pi}{3} r_h^3 = \frac{4}{3}\left(\frac{S^3}{\pi}\right)^{1/2},$$

As $J \to 0$, V and S are not independent.

Thermodynamic Volume

- In $4-D$ the thermodynamic volume evaluates to

$$V = \frac{1}{3}\left(r_h \mathcal{A}_h + \frac{4\pi J^2}{M} \right),$$

where $\mathcal{A}_h$ is area of the event horizon and r_h the radius
(Cvetič *et al.* [1012.2888], BPD [1106.6260]).

- Reverse Isometric inequality,
- As $J \to 0$, $\mathcal{A}_h \to 4\pi r_h^2$ and

$$V \to V_0 := \frac{4\pi}{3} r_h^3 = \frac{4}{3}\left(\frac{S^3}{\pi}\right)^{1/2},$$

As $J \to 0$, V and S are not independent.

Equation of state $(D = 4, \Lambda < 0)$

- Critical point $(J \neq 0)$:

$$(PJ)_{crit} \approx 0.002857\,,\ \left(T\sqrt{J}\right)_{crit} \approx 0.04175\,,\ \left(\frac{V}{J^{3/2}}\right)_{crit} \approx 115.8$$

- Define

$$t := \frac{T - T_c}{T_c}, \qquad v := \frac{V - V_c}{V_c}, \qquad p := \frac{P - P_c}{P_c}.$$

Expand the equation of state about the critical point:

$$p = 2.42t - 0.81tv - 0.21v^3 + o(t^2, tv^2, v^4).$$

cf. Van der Waals gas: $p = 4t - 6tv - \frac{3}{2}v^3 + o(t^2, tv^2, v^4)$.

(Gunasekaran *et al.* [1208.6251], BPD [1209.1272]).

Equation of state $(D = 4, \Lambda < 0)$

- Critical point ($J \neq 0$):

$$(PJ)_{crit} \approx 0.002857\,,\ \left(T\sqrt{J}\right)_{crit} \approx 0.04175\,,\ \left(\frac{V}{J^{3/2}}\right)_{crit} \approx 115.8$$

- Define

$$t := \frac{T - T_c}{T_c}, \qquad v := \frac{V - V_c}{V_c}, \qquad p := \frac{P - P_c}{P_c}.$$

Expand the equation of state about the critical point:

$$p = 2.42t - 0.81tv - 0.21v^3 + o(t^2, tv^2, v^4).$$

cf. Van der Waals gas: $p = 4t - 6tv - \frac{3}{2}v^3 + o(t^2, tv^2, v^4)$.

(Gunasekaran *et al.* [1208.6251], BPD [1209.1272]).

Equation of state $(D = 4, \Lambda < 0)$

- Critical point $(J \neq 0)$:

$$(PJ)_{crit} \approx 0.002857\,,\ \left(T\sqrt{J}\right)_{crit} \approx 0.04175\,,\ \left(\frac{V}{J^{3/2}}\right)_{crit} \approx 115.8$$

- Define

$$t := \frac{T - T_c}{T_c}, \qquad v := \frac{V - V_c}{V_c}, \qquad p := \frac{P - P_c}{P_c}.$$

Expand the equation of state about the critical point:

$$p = 2.42t - 0.81tv - 0.21v^3 + o(t^2, tv^2, v^4).$$

cf. Van der Waals gas: $p = 4t - 6tv - \frac{3}{2}v^3 + o(t^2, tv^2, v^4)$.

(Gunasekaran *et al.* [1208.6251], BPD [1209.1272]).

Equation of state $(D = 4, \Lambda < 0)$

- Critical point ($J \neq 0$):

$$(PJ)_{crit} \approx 0.002857\,,\ \left(T\sqrt{J}\right)_{crit} \approx 0.04175\,,\ \left(\frac{V}{J^{3/2}}\right)_{crit} \approx 115.8$$

- Define

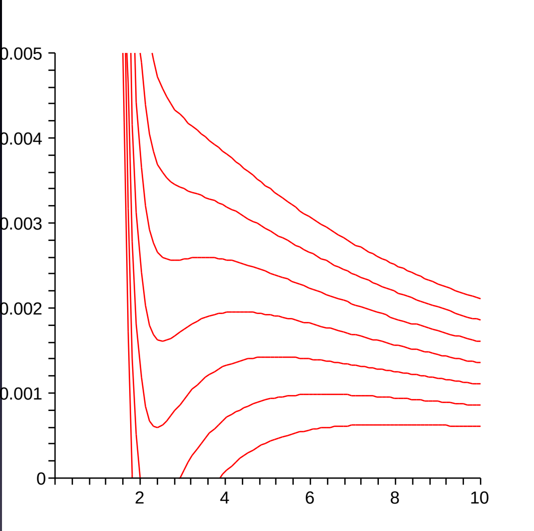
$$t := \frac{T - T_c}{T_c}, \qquad v := \frac{V - V_c}{V_c}, \qquad p := \frac{P - P_c}{P_c}.$$

Expand the equation of state about the critical point:

$$p = 2.42t - 0.81tv - 0.21v^3 + o(t^2, tv^2, v^4).$$

cf. Van der Waals gas: $p = 4t - 6tv - \frac{3}{2}v^3 + o(t^2, tv^2, v^4)$.
(Gunasekaran *et al.* [1208.6251], BPD [1209.1272]).

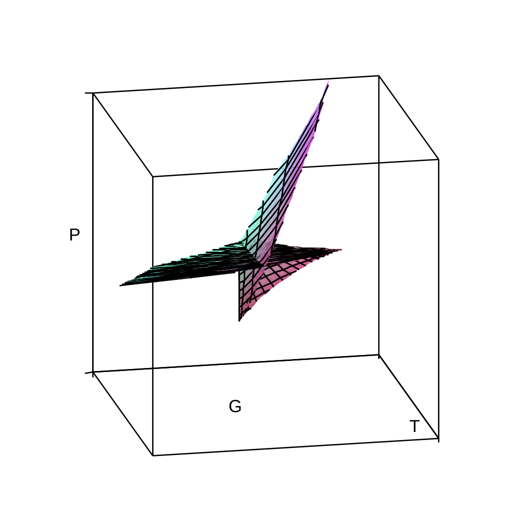
$P - V$ diagram



P as a function of $\left(\frac{3V}{4\pi}\right)^{1/3}$, curves of constant T for $J = 1$.

Gibbs Free Energy

Gibbs Free Energy, $G(T, P, J) = H(S, P, J) - TS$: $(J = 1)$



Critical exponents

- $C_V = T / \left. \frac{\partial T}{\partial S} \right|_{V,J} \propto t^{-\alpha}$;
- At fixed $p < 0$, $v_> - v_< \propto |t|^\beta$;
- Isothermal compressibility, $\kappa_T = -\frac{1}{V}\left(\frac{\partial V}{\partial P}\right)_{T,J} \propto t^{-\gamma}$;
- At $t = 0$, $|p| \propto |v|^\delta$;

Mean Field Exponents

$$\alpha = 0, \qquad \beta = \frac{1}{2}, \qquad \gamma = 1, \qquad \delta = 3.$$

Same as Van der Waals gas

Critical exponents

- $C_V = T / \left.\frac{\partial T}{\partial S}\right|_{V,J} \propto t^{-\alpha}$;
- At fixed $p < 0$, $v_> - v_< \propto |t|^{\beta}$;
- Isothermal compressibility, $\kappa_T = -\frac{1}{V}\left(\frac{\partial V}{\partial P}\right)_{T,J} \propto t^{-\gamma}$;
- At $t = 0$, $|p| \propto |v|^{\delta}$;

Mean Field Exponents

$$\alpha = 0, \qquad \beta = \frac{1}{2}, \qquad \gamma = 1, \qquad \delta = 3.$$

Same as Van der Waals gas

Critical exponents

- $C_V = T / \left. \frac{\partial T}{\partial S} \right|_{V,J} \propto t^{-\alpha}$;
- At fixed $p < 0$, $v_> - v_< \propto |t|^{\beta}$;
- Isothermal compressibility, $\kappa_T = -\frac{1}{V}\left(\frac{\partial V}{\partial P}\right)_{T,J} \propto t^{-\gamma}$;
- At $t = 0$, $|p| \propto |v|^{\delta}$;

Mean Field Exponents

$$\alpha = 0, \qquad \beta = \frac{1}{2}, \qquad \gamma = 1, \qquad \delta = 3.$$

Same as Van der Waals gas

Critical exponents

- $C_V = T / \left.\frac{\partial T}{\partial S}\right|_{V,J} \propto t^{-\alpha}$;
- At fixed $p < 0$, $v_> - v_< \propto |t|^{\beta}$;
- Isothermal compressibility, $\kappa_T = -\frac{1}{V}\left(\frac{\partial V}{\partial P}\right)_{T,J} \propto t^{-\gamma}$;
- At $t = 0$, $|p| \propto |v|^{\delta}$;

Mean Field Exponents

$$\alpha = 0, \qquad \beta = \frac{1}{2}, \qquad \gamma = 1, \qquad \delta = 3.$$

Same as Van der Waals gas

Critical exponents

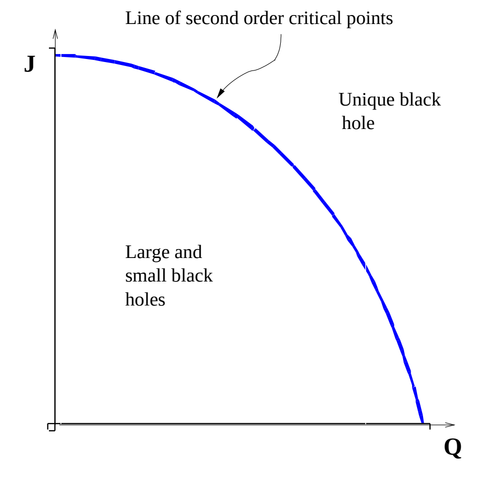
- $C_V = T / \left.\frac{\partial T}{\partial S}\right|_{V,J} \propto t^{-\alpha}$;
- At fixed $p < 0$, $v_> - v_< \propto |t|^\beta$;
- Isothermal compressibility, $\kappa_T = -\frac{1}{V}\left(\frac{\partial V}{\partial P}\right)_{T,J} \propto t^{-\gamma}$;
- At $t = 0$, $|p| \propto |v|^\delta$;

Mean Field Exponents

$$\alpha = 0, \qquad \beta = \frac{1}{2}, \qquad \gamma = 1, \qquad \delta = 3.$$

Same as Van der Waals gas.

Kerr-Reissner-Nordström-AdS



Reissner-Nordström anti-de Sitter ($J \neq 0$, $Q \neq 0$).
(Chamblin, Emparan, Johnston + Myers [hep-th/9902170]; Caldarelli, Gognola+Klemm [hep-th/9908022]; BPD [1209.1272].)

Compressibility

- Asymptotically AdS Myers-Perry in D-dimensions: (rotation parameters a_i, for $\Lambda = 0$, $a_i = \frac{D-2}{2}\frac{J_i}{M}$).

$$V = \frac{1}{D-1}\left(r_h\mathcal{A}_h + \frac{8\pi}{(D-2)}\sum_{i=1} a_i J_i\right)$$

(Cvetič *et al.* [1012.2888], BPD [1106.6260]).

Compressibility

- Adiabatic compressibility: $\kappa = -\frac{1}{V} \left.\frac{\partial V}{\partial P}\right|_{S,J}$.
- Rotating black-hole in D-dimensions (Myers-Perry). Dimensionless angular momenta, $\mathcal{J}_i := \frac{2\pi J_i}{S}$,

  Constraint: $T \geq 0 \Rightarrow \sum_i \frac{1}{1+\mathcal{J}_i^2} \geq \begin{cases} \frac{1}{2} & \text{even } D \\ 1 & \text{odd } D. \end{cases}$

Compressibility, $\Lambda \to 0$

$$\kappa = \frac{16\pi r_h^2}{(D-1)(D-2)^2} \left\{ \frac{(D-2)\sum_i \mathcal{J}_i^4 - (\sum_i \mathcal{J}_i^2)^2}{D-2+\sum_i \mathcal{J}_i^2} \right\},$$

(BPD [1308.5403])

- $0 \leq \kappa < \infty$.
- e.g. 4-D with $P = 0$, $\kappa_{max} = 2.6 \times 10^{-38} \left(\frac{M}{M_\odot}\right)^2 m\, s^2\, kg^{-1}$.

  cf. neutron star, $M \approx M_\odot$, degenerate Fermi gas $\Rightarrow \kappa \approx 10^{-34} m\, s^2\, kg^{-1}$.

Compressibility

- Adiabatic compressibility: $\kappa = -\frac{1}{V}\left.\frac{\partial V}{\partial P}\right|_{S,J}$.
- Rotating black-hole in D-dimensions (Myers-Perry).
  Dimensionless angular momenta, $\mathcal{J}_i := \frac{2\pi J_i}{S}$,
  Constraint: $T \geq 0 \Rightarrow \sum_i \frac{1}{1+\mathcal{J}_i^2} \geq \begin{cases} \frac{1}{2} & \text{even } D \\ 1 & \text{odd } D. \end{cases}$

Compressibility, $\Lambda \to 0$

$$\kappa = \frac{16\pi r_h^2}{(D-1)(D-2)^2}\left\{\frac{(D-2)\sum_i \mathcal{J}_i^4 - (\sum_i \mathcal{J}_i^2)^2}{D-2+\sum_i \mathcal{J}_i^2}\right\},$$

- $0 \leq \kappa < \infty$.
- e.g. 4-D with $P = 0$, $\kappa_{max} = 2.6 \times 10^{[illegible]}\left(\frac{M}{M_\odot}\right)^{[illegible]} m\, s^2\, kg^{-1}$.
  cf. neutron star, $M \approx M_\odot$, degenerate Fermi gas
  $\Rightarrow \kappa \approx 10^{-34} m\, s^2\, kg^{-1}$.

Compressibility

- Adiabatic compressibility: $\kappa = -\frac{1}{V}\left.\frac{\partial V}{\partial P}\right|_{S,J}$.
- Rotating black-hole in D-dimensions (Myers-Perry). Dimensionless angular momenta, $\mathcal{J}_i := \frac{2\pi J_i}{S}$,

  Constraint: $T \geq 0 \Rightarrow \sum_i \frac{1}{1+\mathcal{J}_i^2} \geq \begin{cases} \frac{1}{2} & \text{even } D \\ 1 & \text{odd } D. \end{cases}$

Compressibility, $\Lambda \to 0$

$$\kappa = \frac{16\pi r_h^2}{(D-1)(D-2)^2}\left\{\frac{(D-2)\sum_i \mathcal{J}_i^4 - (\sum_i \mathcal{J}_i^2)^2}{D-2+\sum_i \mathcal{J}_i^2}\right\},$$

(BPD [1308.5403])

- $0 \leq \kappa < \infty$.
- e.g. 4-D with $P = 0$, $\kappa_{max} = 2.6 \times 10^{-38}\left(\frac{M}{M_\odot}\right)^2 m\, s^2\, kg^{-1}$.
  cf. neutron star, $M \approx M_\odot$, degenerate Fermi gas $\Rightarrow \kappa \approx 10^{-34} m\, s^2\, kg^{-1}$.

Compressibility

- Adiabatic compressibility: $\kappa = -\frac{1}{V}\left.\frac{\partial V}{\partial P}\right|_{S,J}$.
- Rotating black-hole in D-dimensions (Myers-Perry). Dimensionless angular momenta, $\mathcal{J}_i := \frac{2\pi J_i}{S}$,

  Constraint: $T \geq 0 \Rightarrow \sum_i \frac{1}{1+\mathcal{J}_i^2} \geq \begin{cases} \frac{1}{2} & \text{even } D \\ 1 & \text{odd } D. \end{cases}$

Compressibility, $\Lambda \to 0$

$$\kappa = \frac{16\pi r_h^2}{(D-1)(D-2)^2}\left\{\frac{(D-2)\sum_i \mathcal{J}_i^4 - (\sum_i \mathcal{J}_i^2)^2}{D-2+\sum_i \mathcal{J}_i^2}\right\},$$

- $0 \leq \kappa < \infty$.
- e.g. 4-D with $P = 0$, $\kappa_{max} = 2.6 \times 10^{-38}\left(\frac{M}{M_\odot}\right)^2 m\, s^2\, kg^{-1}$. *cf.* neutron star, $M \approx M_\odot$, degenerate Fermi gas $\Rightarrow \kappa \approx 10^{-34} m\, s^2\, kg^{-1}$.

Compressibility

- Adiabatic compressibility: $\kappa = -\frac{1}{V}\left.\frac{\partial V}{\partial P}\right|_{S,J}$.
- Rotating black-hole in D-dimensions (Myers-Perry). Dimensionless angular momenta, $\mathcal{J}_i := \frac{2\pi J_i}{S}$,

Constraint: $T \geq 0 \Rightarrow \sum_i \frac{1}{1+\mathcal{J}_i^2} \geq \begin{cases} \frac{1}{2} & \text{even } D \\ 1 & \text{odd } D. \end{cases}$

Compressibility, $\Lambda \to 0$

$$\kappa = \frac{16\pi r_h^2}{(D-1)(D-2)^2}\left\{\frac{(D-2)\sum_i \mathcal{J}_i^4 - (\sum_i \mathcal{J}_i^2)^2}{D-2+\sum_i \mathcal{J}_i^2}\right\},$$

(BPD [1308.5403])

- $0 \leq \kappa < \infty$.
- e.g. 4-D with $P=0$, $\kappa_{max} = 2.6 \times 10^{-38}\left(\frac{M}{M_\odot}\right)^2 m\, s^2\, kg^{-1}$.
  cf. neutron star, $M \approx M_\odot$, degenerate Fermi gas
  $\Rightarrow \kappa \approx 10^{-34} m\, s^2\, kg^{-1}$.

Compressibility

- Adiabatic compressibility: $\kappa = -\frac{1}{V} \left.\frac{\partial V}{\partial P}\right|_{S,J}$.
- Rotating black-hole in D-dimensions (Myers-Perry).
  Dimensionless angular momenta, $\mathcal{J}_i := \frac{2\pi J_i}{S}$,

  Constraint: $T \geq 0 \Rightarrow \sum_i \frac{1}{1+\mathcal{J}_i^2} \geq \begin{cases} \frac{1}{2} & \text{even } D \\ 1 & \text{odd } D. \end{cases}$

Compressibility, $\Lambda \to 0$

$$\kappa = \frac{16\pi r_h^2}{(D-1)(D-2)^2} \left\{ \frac{(D-2)\sum_i \mathcal{J}_i^4 - (\sum_i \mathcal{J}_i^2)^2}{D-2+\sum_i \mathcal{J}_i^2} \right\},$$

(BPD [1308.5403])

- $0 \leq \kappa < \infty$.
- e.g. 4-D with $P=0$, $\kappa_{max} = 2.6 \times 10^{-38} \left(\frac{M}{M_\odot}\right)^2 m\, s^2\, kg^{-1}$;
  cf. neutron star, $M \approx M_\odot$, degenerate Fermi gas
  $\Rightarrow \kappa \approx 10^{-34} m\, s^2\, kg^{-1}$.

Compressibility

- Adiabatic compressibility: $\kappa = -\frac{1}{V}\frac{\partial V}{\partial P}\Big|_{S,J}$.
- Rotating black-hole in D-dimensions (Myers-Perry). Dimensionless angular momenta, $\mathcal{J}_i := \frac{2\pi J_i}{S}$,

  Constraint: $T \geq 0 \Rightarrow \sum_i \frac{1}{1+\mathcal{J}_i^2} \geq \begin{cases} \frac{1}{2} & \text{even } D \\ 1 & \text{odd } D. \end{cases}$

Compressibility, $\Lambda \to 0$

$$\kappa = \frac{16\pi r_h^2}{(D-1)(D-2)^2}\left\{\frac{(D-2)\sum_i \mathcal{J}_i^4 - (\sum_i \mathcal{J}_i^2)^2}{D-2+\sum_i \mathcal{J}_i^2}\right\},$$

(BPD [1308.5403])

- $0 \leq \kappa < \infty$.
- e.g. 4-D with $P = 0$, $\kappa_{max} = 2.6 \times 10^{-38}\left(\frac{M}{M_\odot}\right)^2 m\, s^2\, kg^{-1}$.

  cf. neutron star, $M \approx M_\odot$, degenerate Fermi gas $\Rightarrow \kappa \approx 10^{-34} m\, s^2\, kg^{-1}$.

Compressibility

- Adiabatic compressibility: $\kappa = -\frac{1}{V}\left.\frac{\partial V}{\partial P}\right|_{S,J}$.
- Rotating black-hole in D-dimensions (Myers-Perry).
  Dimensionless angular momenta, $\mathcal{J}_i := \frac{2\pi J_i}{S}$,

  Constraint: $T \geq 0 \Rightarrow \sum_i \frac{1}{1+\mathcal{J}_i^2} \geq \begin{cases} \frac{1}{2} & \text{even } D \\ 1 & \text{odd } D. \end{cases}$

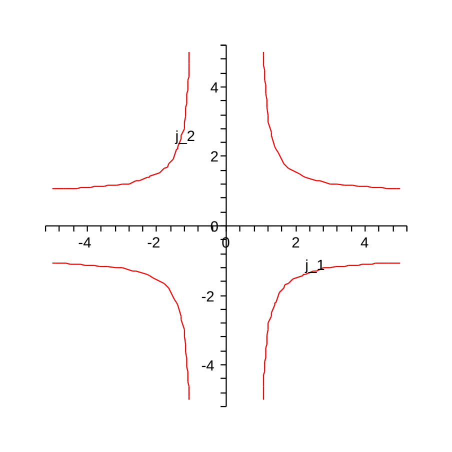
Compressibility, $\Lambda \to 0$

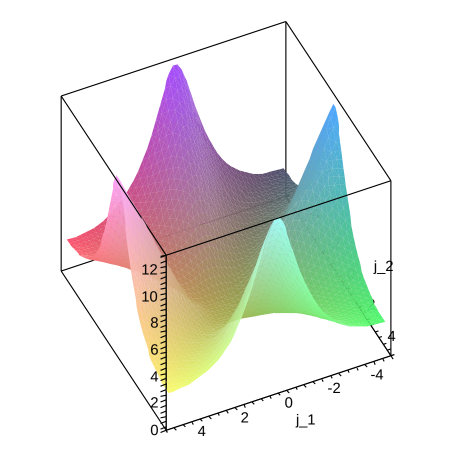
$$\kappa = \frac{16\pi r_h^2}{(D-1)(D-2)^2}\left\{\frac{(D-2)\sum_i \mathcal{J}_i^4 - (\sum_i \mathcal{J}_i^2)^2}{D-2+\sum_i \mathcal{J}_i^2}\right\},$$

(BPD [1308.5403])

- $0 \leq \kappa < \infty$.
- e.g. 4-D with $P = 0$, $\kappa_{max} = 2.6 \times 10^{-38}\left(\frac{M}{M_\odot}\right)^2 m\, s^2\, kg^{-1}$.
  cf. neutron star, $M \approx M_\odot$, degenerate Fermi gas
  $\Rightarrow \kappa \approx 10^{-34} m\, s^2\, kg^{-1}$.

Example of compressibility, $D = 6$, $SO(5)$: J_1, J_2,





Speed of sound

- Define $\rho := \frac{M}{V}$, then the thermodynamic speed of sound is

$$c_s^2 = \left.\frac{\partial P}{\partial \rho}\right|_{S,J}, \qquad \Rightarrow \qquad {c_s}^{-2} = 1 + \kappa\rho.$$

Speed of sound

$$c_s^2 = \frac{1}{(D-2)} \frac{\left(D - 2 + \sum_i \mathcal{J}_i^2\right)^2}{\left(D - 2 + 2\sum_i \mathcal{J}_i^2 + \sum_i \mathcal{J}_i^4\right)}.$$

- $\frac{1}{D-2} \le c_s^2 \le 1.$

Speed of sound

- Define $\rho := \frac{M}{V}$, then the thermodynamic speed of sound is

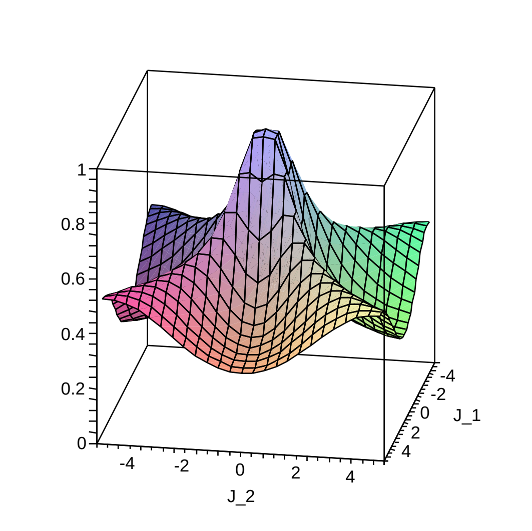
$$c_s^2 = \left.\frac{\partial P}{\partial \rho}\right|_{S,J}, \qquad \Rightarrow \qquad c_s^{-2} = 1 + \kappa\rho.$$

Speed of sound

$$c_s^2 = \frac{1}{(D-2)} \frac{\left(D - 2 + \sum_i \mathcal{J}_i^2\right)^2}{\left(D - 2 + 2\sum_i \mathcal{J}_i^2 + \sum_i \mathcal{J}_i^4\right)}.$$

- $\frac{1}{D-2} \leq c_s^2 \leq 1$.

c_s^2 in 6-dimensions:



Asymptotically de Sitter

BPD, D. Kastor, D. Kubiznak, R.B. Mann and J. Traschen [1301.5926]

- $P = -\frac{\Lambda}{8\pi} < 0$.
- Two event horizons: black hole r_{bh}; cosmological, r_c.
- Two different temperatures, $T_{bh} \neq T_c$, in general.
- $M(S_{bh}, P, J) = M(S_c, P, J)$.
  $V_{bh} = \left.\frac{\partial M}{\partial P}\right|_{S_{bh},J}$, $V_c = \left.\frac{\partial M}{\partial P}\right|_{S_c,J}$.
- $V_{bh} = \frac{r_{bh}A_{bh}}{3} + \frac{4\pi}{3}\frac{J^2}{M}, \qquad V_c = \frac{r_c A_c}{3} + \frac{4\pi}{3}\frac{J^2}{M}$.
- For fixed V_c the cosmological horizon entropy, $S_c = \frac{A_c}{4}$, is maximized by Schwarzschild- de Sitter space-time
- Volume between horizons: $V = V_c - V_{bh} = \frac{r_c A_c}{3} - \frac{r_{bh}A_{bh}}{3}$.
- For fixed V the total entropy, $S = S_{bh} + S_c$, is minimized if there is no black hole.

Asymptotically de Sitter

BPD, D. Kastor, D. Kubiznak, R.B. Mann and J. Traschen [1301.5926]

- $P = -\frac{\Lambda}{8\pi} < 0$.
- Two event horizons: black hole r_{bh}; cosmological, r_c.
- Two different temperatures, $T_{bh} \neq T_c$, in general.
- $M(S_{bh}, P, J) = M(S_c, P, J)$.
  $V_{bh} = \frac{\partial M}{\partial P}\Big|_{S_{bh},J}$, $V_c = \frac{\partial M}{\partial P}\Big|_{S_c,J}$.
- $V_{bh} = \frac{r_{bh}A_{bh}}{3} + \frac{4\pi}{3}\frac{J^2}{M}, \qquad V_c = \frac{r_c A_c}{3} + \frac{4\pi}{3}\frac{J^2}{M}$.
- For fixed V_c the cosmological horizon entropy, $S_c = \frac{A_c}{4}$, is maximized by Schwarzschild- de Sitter space-time
- Volume between horizons: $V = V_c - V_{bh} = \frac{r_c A_c}{3} - \frac{r_{bh}A_{bh}}{3}$.
- For fixed V the total entropy, $S = S_{bh} + S_c$, is minimized if there is no black hole.

Asymptotically de Sitter

BPD, D. Kastor, D. Kubiznak, R.B. Mann and J. Traschen [1301.5926]

- $P = -\frac{\Lambda}{8\pi} < 0$.
- Two event horizons: black hole r_{bh}; cosmological, r_c.
- Two different temperatures, $T_{bh} \neq T_c$, in general.
- $M(S_{bh}, P, J) = M(S_c, P, J)$.
  $V_{bh} = \left.\frac{\partial M}{\partial P}\right|_{S_{bh},J}$, $V_c = \left.\frac{\partial M}{\partial P}\right|_{S_c,J}$.
- $V_{bh} = \frac{r_{bh}A_{bh}}{3} + \frac{4\pi}{3}\frac{J^2}{M}$, $\quad V_c = \frac{r_c A_c}{3} + \frac{4\pi}{3}\frac{J^2}{M}$.
- For fixed V_c the cosmological horizon entropy, $S_c = \frac{A_c}{4}$, is maximized by Schwarzschild- de Sitter space-time
- Volume between horizons: $V = V_c - V_{bh} = \frac{r_c A_c}{3} - \frac{r_{bh}A_{bh}}{3}$.
- For fixed V the total entropy, $S = S_{bh} + S_c$, is minimized if there is no black hole.

Asymptotically de Sitter

BPD, D. Kastor, D. Kubiznak, R.B. Mann and J. Traschen [1301.5926]

- $P = -\frac{\Lambda}{8\pi} < 0$.
- Two event horizons: black hole r_{bh}; cosmological, r_c.
- Two different temperatures, $T_{bh} \neq T_c$, in general.
- $M(S_{bh}, P, J) = M(S_c, P, J)$.
  $V_{bh} = \left.\frac{\partial M}{\partial P}\right|_{S_{bh},J}$, $V_c = \left.\frac{\partial M}{\partial P}\right|_{S_c,J}$.
- $V_{bh} = \frac{r_{bh}A_{bh}}{3} + \frac{4\pi}{3}\frac{J^2}{M}$, $\qquad V_c = \frac{r_c A_c}{3} + \frac{4\pi}{3}\frac{J^2}{M}$.
- For fixed V_c the cosmological horizon entropy, $S_c = \frac{A_c}{4}$, is maximized by Schwarzschild- de Sitter space-time
- Volume between horizons: $V = V_c - V_{bh} = \frac{r_c A_c}{3} - \frac{r_{bh}A_{bh}}{3}$.
- For fixed V the total entropy, $S = S_{bh} + S_c$, is minimized if there is no black hole.

Asymptotically de Sitter

BPD, D. Kastor, D. Kubiznak, R.B. Mann and J. Traschen [1301.5926]

- $P = -\frac{\Lambda}{8\pi} < 0$.
- Two event horizons: black hole r_{bh}; cosmological, r_c.
- Two different temperatures, $T_{bh} \neq T_c$, in general.
- $M(S_{bh}, P, J) = M(S_c, P, J)$.
  $V_{bh} = \left.\frac{\partial M}{\partial P}\right|_{S_{bh},J}$, $V_c = \left.\frac{\partial M}{\partial P}\right|_{S_c,J}$.
- $V_{bh} = \frac{r_{bh}A_{bh}}{3} + \frac{4\pi}{3}\frac{J^2}{M}, \qquad V_c = \frac{r_c A_c}{3} + \frac{4\pi}{3}\frac{J^2}{M}$.
- For fixed V_c the cosmological horizon entropy, $S_c = \frac{A_c}{4}$, is maximized by Schwarzschild- de Sitter space-time
- Volume between horizons: $V = V_c - V_{bh} = \frac{r_c A_c}{3} - \frac{r_{bh}A_{bh}}{3}$.
- For fixed V the total entropy, $S = S_{bh} + S_c$, is minimized if there is no black hole.

Asymptotically de Sitter

BPD, D. Kastor, D. Kubiznak, R.B. Mann and J. Traschen [1301.5926]

- $P = -\frac{\Lambda}{8\pi} < 0$.
- Two event horizons: black hole r_{bh}; cosmological, r_c.
- Two different temperatures, $T_{bh} \neq T_c$, in general.
- $M(S_{bh}, P, J) = M(S_c, P, J)$.
  $V_{bh} = \left.\frac{\partial M}{\partial P}\right|_{S_{bh},J}$, $V_c = \left.\frac{\partial M}{\partial P}\right|_{S_c,J}$.
- $V_{bh} = \frac{r_{bh}A_{bh}}{3} + \frac{4\pi}{3}\frac{J^2}{M}$, $\qquad V_c = \frac{r_c A_c}{3} + \frac{4\pi}{3}\frac{J^2}{M}$.
- For fixed V_c the cosmological horizon entropy, $S_c = \frac{A_c}{4}$, is maximized by Schwarzschild- de Sitter space-time $(J = 0)$.
- Volume between horizons: $V = V_c - V_{bh} = \frac{r_c A_c}{3} - \frac{r_{bh}A_{bh}}{3}$.
- For fixed V the total entropy, $S = S_{bh} + S_c$, is minimized if there is no black hole.

Asymptotically de Sitter

BPD, D. Kastor, D. Kubiznak, R.B. Mann and J. Traschen [1301.5926]

- $P = -\frac{\Lambda}{8\pi} < 0.$
- Two event horizons: black hole r_{bh}; cosmological, r_c.
- Two different temperatures, $T_{bh} \neq T_c$, in general.
- $M(S_{bh}, P, J) = M(S_c, P, J)$.
  $V_{bh} = \left.\frac{\partial M}{\partial P}\right|_{S_{bh},J}, \; V_c = \left.\frac{\partial M}{\partial P}\right|_{S_c,J}.$
- $V_{bh} = \frac{r_{bh}A_{bh}}{3} + \frac{4\pi}{3}\frac{J^2}{M}, \qquad V_c = \frac{r_c A_c}{3} + \frac{4\pi}{3}\frac{J^2}{M}.$
- For fixed V_c the cosmological horizon entropy, $S_c = \frac{A_c}{4}$, is maximized by Schwarzschild- de Sitter space-time $(J = 0)$.
- Volume between horizons: $V = V_c - V_{bh} = \frac{r_c A_c}{3} - \frac{r_{bh}A_{bh}}{3}.$
- For fixed V the total entropy, $S = S_{bh} + S_c$, is minimized if there is no black hole.

Asymptotically de Sitter

BPD, D. Kastor, D. Kubiznak, R.B. Mann and J. Traschen [1301.5926]

- $P = -\frac{\Lambda}{8\pi} < 0$.
- Two event horizons: black hole r_{bh}; cosmological, r_c.
- Two different temperatures, $T_{bh} \neq T_c$, in general.
- $M(S_{bh}, P, J) = M(S_c, P, J)$.
  $V_{bh} = \frac{\partial M}{\partial P}\Big|_{S_{bh},J}$, $V_c = \frac{\partial M}{\partial P}\Big|_{S_c,J}$.
- $V_{bh} = \frac{r_{bh}A_{bh}}{3} + \frac{4\pi}{3}\frac{J^2}{M}, \qquad V_c = \frac{r_c A_c}{3} + \frac{4\pi}{3}\frac{J^2}{M}$.
- For fixed V_c the cosmological horizon entropy, $S_c = \frac{A_c}{4}$, is maximized by Schwarzschild- de Sitter space-time $(J = 0)$.
- Volume between horizons: $V = V_c - V_{bh} = \frac{r_c A_c}{3} - \frac{r_{bh}A_{bh}}{3}$.
- For fixed V the total entropy, $S = S_{bh} + S_c$, is minimized if there is no black hole.

Conclusions

- $\Lambda \neq 0 \quad \Rightarrow \quad P\,dV$ term in black hole 1st law.
- Black hole mass is identified with **enthalpy**, $H(S, P, J_i)$:

$$\begin{aligned} dM &= dH = T\,dS + V\,dP + \Omega_i J_i\,, \\ dU &= T\,dS - P\,dV + \Omega_i J_i\,. \end{aligned}$$

- Hawking temperature: $T = \left(\frac{\partial H}{\partial S}\right)_P$.
- "Thermodynamic" volume: $V = \left(\frac{\partial H}{\partial P}\right)_T$.
- PdV term affects Penrose processes — more efficient in asymptotically AdS space-times.
- $D = 4$: Van der Waals type equation of state.
- Compressibility, $0 \leq \kappa < \infty$, with $\kappa \to \infty$ for some $J_i \to \infty$.
- Instability of ultra-spinning black-holes.
- Speed of sound: $\frac{1}{D-2} \leq c_s^2 \leq 1$.

Ripples on the horizon

- Ultra-spinning black-holes are unstable (Emparan + Myers hep-th/0308056).
- $\Lambda = 0$, $D \geq 6$: (Emparan, Harmark, Niarchos, Obers + Rodriguez [hep-th:0708.2181]).

Ripples on the horizon

- Ultra-spinning black-holes are unstable (Emparan + Myers hep-th/0308056).
- $\Lambda = 0$, $D \geq 6$: (Emparan, Harmark, Niarchos, Obers + Rodriguez [hep-th:0708.2181]).